Multivariate isotonic time series regression

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(with **Konstantinos Fokianos, Anne Leucht**)

Motivation: Estimation of integer-valued time series

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nonparametric (time series) regression:

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particular challenge: explanatory variables not "regularly" distributed
$\Longrightarrow$ suitability of standard nonparametric methods (kernel,...) not clear

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... linear models, e.g. integer-valued AR(p):

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$\Longrightarrow f\colon \mathbb{R}^p \to \mathbb{R}$ isotonic (non-decreasing in each component)

Isotonic regression: Classical least squares estimator

$$Y_t = f(I_t) + \varepsilon_t, \quad t = 1, \ldots, n,$$

- I_t d-dimensional information variable
- $E(\varepsilon_t \mid I_t) = 0$ a.s.
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Isotonic least squares estimator:

$$\widetilde{f}_n \in \arg \min_{g \text{ isotonic}} \sum_{t=1}^{n} (Y_t - g(I_t))^2$$

Isotonic regression: Classical least squares estimator

Advantages of $\widetilde{f}_n$:

- no bandwidth choice necessary
- $d = 1$: irregular distribution of I_t doesn't harm,
  $\int |\widetilde{f}_n(x) - f(x)| \, dP^{I_t}(x)$ converges with optimal rate $n^{-1/3}$

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Appropriateness of isotonicity condition?

- many parametric models for integer-valued time series produce isotonic conditional mean functions
- applications in many areas:
  biology, medicine, statistics, psychology, genetics
  (see. e.g. Luss, Rosset & Shahar (*Ann. Statist.*, 2012))

Classical isotonic least squares estimator: Details

Alternative representation of $\widetilde{f}_n$:

If $x \in \{X_1, \ldots, X_n\}$, then (Brunk, 1955)

$$\begin{aligned}\widetilde{f}_n(x) &= \max_{U:\, x \in U} \min_{L:\, x \in L} \mathrm{Av}_Y(L \cap U) \\ &= \min_{L:\, x \in L} \max_{U:\, x \in U} \mathrm{Av}_Y(L \cap U),\end{aligned}$$

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- L lower set
  $(y \in L,\ z \preceq y \Rightarrow z \in L)$
- U upper set
  $(y \in U,\ z \succeq y \Rightarrow z \in U)$
- $z \preceq y$ means $z_i \le y_i$
  $\forall i = 1, \dots, d$

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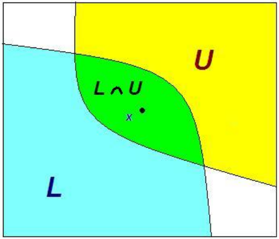

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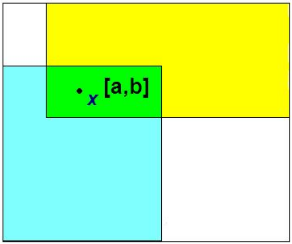

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Proposed estimator:

$$\widehat{f}_n(x) = \frac{\overline{f}_n(x) + \overline{\overline{f}}_n(x)}{2}$$

Main result: What could be expected?

- $f\colon\ [0,1]^d \longrightarrow \mathbb{R}$ isotonic + bounded (+ differentiable)
  $\Rightarrow \int \sum_{i=1}^d |\partial_i f| < \infty$,
  i.e. degree of smoothness $\beta = 1$
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- No guarantee for a good pointwise behavior!
- Since smoothness is measured in L_1, we consider L_1-loss, $\int |\widehat{f_n} - f| \, dP^{I_1}$

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- $(I_t)_{t \in \mathbb{N}}$ stationary, I_t has a continuous distribution,
- $E(\varepsilon_t \mid I_1, \ldots, I_t, \varepsilon_1, \ldots, \varepsilon_{t-1}) = 0 \quad a.s.$,
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$$\underline{c}\, Q_1 \otimes \cdots \otimes Q_d(\cdot) \le P(I_t \in \cdot) \le \overline{c}\, Q_1 \otimes \cdots \otimes Q_d(\cdot),$$

for some $0 < \underline{c} \le \overline{c} < \infty$, $Q_1, \dots, Q_d$ probability distributions

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  for some $0 < \underline{c} \le \overline{c} < \infty$, $Q_1, \ldots, Q_d$ probability distributions
- $(I_t)_t$ strong $(\alpha-)$ mixing, $\sum_{r=1}^{\infty} r^{d-1/2} \alpha(r) < \infty$

Main result: A special case

- information variable I_t has a density, bounded from zero on $[0,1]^q$
- $h_n = n^{-1/(2+d)}$ (≈ optimal bandwidth)
- $D_n = [h_n, 1 - h_n]^d$

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Theorem 3.1.

$$\int_{D_n} \left| \widehat{f_n}(z) - f(z) \right| dP^{I_1}(z) = O_P\left(n^{-1/(2+d)} \right).$$

Proof of Theorem 3.1

$$\int_{D_n} \left|\widehat{f}_n(z) - f(z)\right| \, dP^{I_1}(z)$$
$$= \int_{D_n} \left(\widehat{f}_n(z) - f(z)\right)_+ dP^{I_1}(z) + \int_{D_n} \left(\widehat{f}_n(z) - f(z)\right)_- dP^{I_1}(z)$$

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For appropriate $b_n = b_n(z) \geq z$:

$$\left(\widehat{f}_n(z) - f(z)\right)_+$$
$$\leq \left(\max_{a:\, a \leq z} \mathrm{Av}_Y([\![a, b_n]\!]) - f(z)\right)_+$$

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$$\leq \left(f(b_n) - f(z)\right) + \left(\max_{a:\, a \leq z} \mathrm{Av}_\varepsilon([\![a, b_n]\!])\right)_+$$

Proof of Theorem 3.1 (cont'd)

Since $\sup_z f(z) - \inf_z f(z) < \infty$,

$$\int_{D_n} (f(b_n) - f(z)) \, dP^{I_t}(z) = O\left(n^{-1/(2+d)}\right).$$

Theorem 1 of Bickel & Wichura (1971) (on fluctuations of an empirical process...):

$$\int_{D_n} \left(\max_{a:\, a \leq z} \mathrm{Av}_\varepsilon([\![a, b_n]\!]) \right)_+ dP^{I_t}(z) = O_P\left(n^{-1/(2+d)}\right).$$

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- $f\colon \mathbb{R}^{p+q} \to \mathbb{R}$ isotonic and bounded
- $\widehat{f}_n$ (modified) nonparametric isotonic estimator
- no choice of smoothing parameter required
- rate of convergence: $n^{-1/(1+q)}$ (optimal for degree of smothness $\beta = 1$, dimension q)