Statistical inference for Vasicek-type model driven by Hermite processes

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Content

Central and non-central limit theorems

General framework:

- Let $X = (X_i)_{i\in\mathbb{N}}$ be a centered stationary Gaussian sequence with variance 1, and $\rho(i-j) = E[X_iX_j]$ be its covariance kernel.
- Let $g \in L^2(\mathbb{R}, \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx)$ such that $\int_{\mathbb{R}} g(x)e^{-\frac{x^2}{2}}dx = 0$, i.e.,

$$g(x) = \sum_{k=q}^{\infty} c_k H_k(x)$$

where $q \geq 1, a_q \neq 0$. Here q is called the Hermite rank of g and H_k is the k-th Hermite polynomial
$H_1(x) = x, H_2(x) = x^2 - 1, H_3(x) = x^3 - 3x, \ldots.$

Question:

$$\sum_{i=1}^{[Nt]} g(X_i) \longrightarrow ??? \qquad \text{as } n \to \infty, t \geq 0$$

Central and non-central limit theorems

1. Central limit theorem (Breuer-Major): When X has short-range dependence (i.e., $\sum_{n\in\mathbb{Z}} |\rho(n)|^q < \infty$), then for all $q \geq 1$:

$$\frac{1}{\sqrt{N}} \sum_{i=1}^{[Nt]} g(X_i) \xrightarrow{f.d.d} \text{Brownian motion}$$

2. Non-central limit theorem (Dobrushin-Major-Taqqu): When X has long-range dependence (i.e., $\sum_{n\in\mathbb{Z}} |\rho(n)|^q = \infty$). Particular, $\rho(n) = E[X_0 X_n] = n^{2H_0-2}L(n)$ for some $H_0 \in (1 - \frac{1}{2q}, 1)$ and L a slowly varying function. Then,

$$\frac{1}{N^H} \sum_{i=1}^{[Nt]} g(X_i) \xrightarrow{f.d.d} \text{Hermite process } Z^{q,H}$$

We have CLT for $q = 1$, limit behaviour $Z^{1,H}$ is the well-known fractional Brownian motion

Fractional Brownian motion

Definition

Let $H \in (0,1]$. A fractional Brownian motion (fBm in short) of Hurst parameter H is a centered continuous Gaussian process $B^H = (B^H_t)_{t\geq 0}$ with covariance function

$$E[B^H_t B^H_s] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

- If $H = \frac{1}{2}$, that means $B^{\frac{1}{2}}$, is a classical Brownian motion.
- If $H = 1$, then $B^H_t = tB^H_1$ almost surely for all $t \geq 0$.

Since $H = 1$ is trivial, we always assume $H \in (0,1)$.

I. Nourdin (2012): Selected aspects of fractional Brownian motion. Bocconi and Springer Series, 4. Springer.

Basic properties

Proposition

(i) *Self-similarity:* for all $a > 0, (a^{-H} B^H_{at})_{t\geq 0} \overset{law}{=} (B^H_t)_{t\geq 0}$

(ii) *Stationary increments:* for all $h > 0, (B^H_{t+h} - B^H_h)_{t\geq 0} \overset{law}{=} (B^H_t)_{t\geq 0}$

(iii) *Long-range dependence $H > \frac{1}{2}$:* $\sum_{n=1}^{\infty} E[B^H_1(B^H_{n+1} - B^H_n)] = \infty$

(iv) *Hölder continuity:* the sample paths of B^H are α-Hölder continuous on each compact set for any $\alpha \in (0, H)$.

(v) fBm is neither a semimartingale nor a Markov process, except when its Hurst parameter is $\frac{1}{2}$ (that means, Brownian motion)

(vi) fBm has the form of a Volterra process, i.e., can be represented as $B^H_t = \int_0^t K_H(t,s)dW_s$, where $W = (W_t)_{t\geq 0}$ is classical Brownian motion and K_H is an explicit square integrable kernel.

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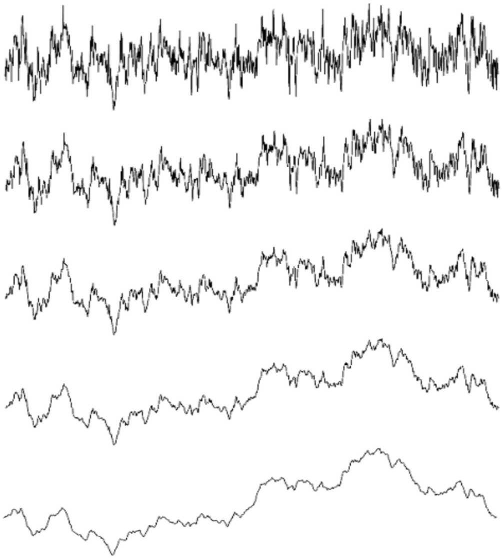

FIGURE 1. Paths of fBm for different values of H.

Hermite processes in a nutshell

V. Pipiras, M. Taqqu (2017): *Long-range dependence and self-similarity*. Cambridge series in statistical and probabilistic mathematics. Cambridge University Press, Cambridge.

C.A. Tudor (2013): *Analysis of variations for self-similar processes: A stochastic calculus approach*. Probability and its Applications. *Springer*.

Where do Hermite processes come from ?

Hermite processes appeared for the first time in a non-CLT proved by Taqqu (1975, 1979) and Dobrushin and Major (1979).

Recall, assume that:

- g belongs to $L^2(\mathbb{R}, \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx)$ and satisfies $\int_{\mathbb{R}} g(x) e^{-\frac{x^2}{2}} dx = 0$ (that is, $g(x) = \sum_{k=q}^{\infty} c_k H_k(x)$ where H_k denotes the k-th Hermite polynomial and q is the Hermite rank of g)
- $X = (X_i)_{i \in \mathbb{Z}}$ is a centered stationary Gaussian seq. with variance 1, satisfying $E[X_0 X_n] = n^{2H_0 - 2} L(n)$ for some $H_0 \in (1 - \frac{1}{2q}, 1)$ and L a slowly varying function.

Then,

$$\frac{1}{N^H} \sum_{i=1}^{[Nt]} g(X_i) \xrightarrow{f.d.d} \text{"Hermite process } Z_t^{q,H}\text{"}.$$

Here $H = q(H_0 - 1) + 1$ belongs to $(\frac{1}{2}, 1)$ for all $q \geq 1$.

Multiple Wiener-Itô integrals and Wiener chaos

Let $B = B(h), h \in L^2(\mathbb{R})$ be a Brownian field defined on a probability space $(\Omega, \mathcal{F}, P)$ satisfying $E[B(h)B(g)] = \langle h, g\rangle_{L^2(\mathbb{R})}$, $\forall h, g \in L^2(\mathbb{R})$.

For every $q \geq 1$, the qth Wiener chaos $\mathcal{H}_q^B$ is defined as the closed linear subspace of $L^2(\Omega)$ generated by the family of random variables

$$\{H_q(B(h)), h \in L^2(\mathbb{R}), \|h\|_{L^2(\mathbb{R})} = 1\}$$

The mapping $I_q^B(h^{\otimes q}) = H_q(B(h))$ can be extended to a linear isometry between $L_s^2(\mathbb{R}^q)$ and the qth Wiener chaos $\mathcal{H}_q^B$. When $f \in L_s^2(\mathbb{R}^q)$, the random variable $I_q^B(f)$ is called the multiple Wiener-Itô integral of f of order q. One may write

$$I_q^B(f) = \int_{\mathbb{R}^q} f(\xi_1, \ldots, \xi_p) dB_{\xi_1} \ldots dB_{\xi_p}$$

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Multiple Wiener-Itô integrals

For any $f \in L^2_s(\mathbb{R}^p)$ and $g \in L^2_s(\mathbb{R}^q)$, we have

- Orthogonality - Isometry:

$$E[I_p^B(f) I_q^B(g)] = \begin{cases} p! \langle f, g \rangle_{L^2(\mathbb{R}^p)} & \text{if } p = q \\ 0 & \text{if } p \neq q. \end{cases}$$

- Hypercontractivity: (All L^k-norms are equivalent)

$$E[|I_p^B(f)|^k]^{1/k} \leq (k-1)^{p/2} E[|I_p^B(f)|^2]^{1/2} \text{ for any } k \in [2, \infty).$$

See D. Nualart (2006): The Malliavin calculus and related topics.

Hermite process viewed as a multiple Wiener-Itô integral

Definition

The Hermite process $(Z_t^{q,H})_{t\geq 0}$ of order $q \geq 1$ and self-similarity parameter $H \in (\frac{1}{2}, 1)$ is defined as

$$Z_t^{q,H} = c(H,q) \int_{\mathbb{R}^q} \left(\int_0^t \prod_{j=1}^{q} (s-\xi_j)_+^{H_0 - \frac{3}{2}} ds \right) dB_{\xi_1} \dots dB_{\xi_q}, \quad (1)$$

where $(B_t)_{t\in\mathbb{R}}$ is a two-sided standard Brownian motion and

$$c(H,q) = \sqrt{\frac{H(2H-1)}{q!\beta^q(H_0 - \frac{1}{2}, 2-2H_0)}}; H_0 = 1 + \frac{H-1}{q} \in \left(1 - \frac{1}{2q}, 1\right). \quad (2)$$

Here $c(H,q)$ is calculated to ensure that $E[(Z_1^{q,H})^2] = 1$.

Hermite random variable

Hermite process of order $q = 1$ is nothing but the fractional Brownian motion. fBm is the only Hermite process to be Gaussian (and that one could have defined for $H \leq \frac{1}{2}$ as well).

Definition

A random variable which has the same law as $Z_1^{q,H}$ is called a Hermite random variable of order q and parameter H.

Hermite process of order $q = 2$ is called the Rosenblatt process. Denoted by $R^H = (R_t^H)_{t \geq 0}$.

Definition

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Properties of Hermite processes

The Hermite process $Z^{q,H}$ shares many properties with the fBm (corresponding to $q = 1$), EXCEPT GAUSSIANITY (for $q \geq 2$).

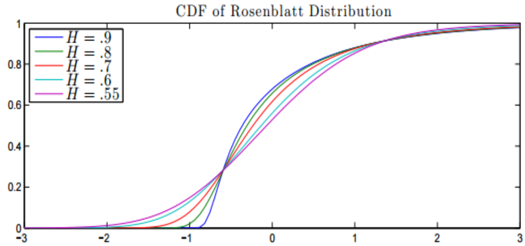
- Self-similarity
  For all $c > 0$, $(Z^{q,H}_{ct})_{t \geq 0} \overset{law}{=} (c^H Z^{q,H}_t)_{t \geq 0}$.
- Stationarity of increments
  For any $h > 0$, $(Z^{q,H}_{t+h} - Z^{q,H}_h)_{t \geq 0} \overset{law}{=} (Z^{q,H}_t)_{t \geq 0}$.
- Covariance function
  For all $s, t \geq 0$, $E[Z^{q,H}_t Z^{q,H}_s] = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H})$.
- Long-range dependence $\sum_{n \geq 1} |E[Z^{q,H}_1 (Z^{q,H}_{n+1} - Z^{q,H}_n)]| = \infty$.
- Hölder continuity $Z^{q,H}$ admits a version with Hölder continuous sample paths of any order $\beta \in (0, H)$ on any compact interval.
- Finite moments
  For every $p \geq 1, t \geq 0$, $E[|Z^{q,H}_t|^p] \leq C_{p,q} t^{pH}$.

Small digression: a mysterious conjecture on Rosenblatt distribution

Unfortunately, the explicit distribution of Rosenblatt random variable is not known (only numerical approximations are available).

Mysterious conjecture (Taqqu and Veillette 2013): whatever the value of H, it seems that

$$P(R_1^H \leq -0.6256) = 0.2658,$$
$$P(R_1^H \leq 1.3552) = 0.9123.$$



Wiener integrals with respect to Hermite process

- $\int_{\mathbb{R}} f(u) dZ_u^{q,H}$ is well-defined for any $f : \mathbb{R} \to \mathbb{R}$ such that

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |f(u)f(v)||u-v|^{2H-2} du dv < \infty.$$

- Isometry $f \longmapsto \int_{\mathbb{R}} f(u) dZ_u^{q,H}$:

$$E\left[\int_{\mathbb{R}} f(u) dZ_u^{q,H} \int_{\mathbb{R}} g(v) dZ_v^{q,H}\right]$$
$$= H(2H-1) \int_{\mathbb{R}} \int_{\mathbb{R}} f(u)g(v)|u-v|^{2H-2} du dv.$$

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Wiener integrals with respect to Hermite process

It can be expressed as a multiple Wiener-Itô integral

$$\int_{\mathbb{R}} f(u) dZ_u^{q,H} = c(H,q) \int_{\mathbb{R}^q} \left(\int_{\mathbb{R}} f(u) \prod_{j=1}^{q} (u - \xi_j)_+^{H_0 - \frac{3}{2}} du \right) dB_{\xi_1} \dots dB_{\xi_q},$$

with $c(H,q)$ and H_0 given in (2).

Choosing $f(u) = \mathbf{1}_{[0,t]}(u)$, then

$$\int_{\mathbb{R}} \mathbf{1}_{[0,t]}(u) dZ_u^{q,H}$$

$$= c(H,q) \int_{\mathbb{R}^q} \left(\int_{\mathbb{R}} \mathbf{1}_{[0,t]}(u) \prod_{j=1}^{q} (u - \xi_j)_+^{H_0 - \frac{3}{2}} du \right) dB_{\xi_1} \dots dB_{\xi_q}$$

$$= c(H,q) \int_{\mathbb{R}^q} \left(\int_0^t \prod_{j=1}^{q} (u - \xi_j)_+^{H_0 - \frac{3}{2}} du \right) dB_{\xi_1} \dots dB_{\xi_q}$$

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$$\begin{aligned} &\int_{\mathbb{R}} \mathbf{1}_{[0,t]}(u) dZ_u^{q,H} \\ &= c(H,q) \int_{\mathbb{R}^q} \left(\int_{\mathbb{R}} \mathbf{1}_{[0,t]}(u) \prod_{j=1}^{q} (u - \xi_j)_+^{H_0 - \frac{3}{2}} du \right) dB_{\xi_1} \dots dB_{\xi_q} \\ &= c(H,q) \int_{\mathbb{R}^q} \left(\int_0^t \prod_{j=1}^{q} (u - \xi_j)_+^{H_0 - \frac{3}{2}} du \right) dB_{\xi_1} \dots dB_{\xi_q} \\ &= Z_t^{q,H} \end{aligned}$$

Vasicek-type model driven by Hermite processes - Motivation

- The (non-stationary) fractional Vasicek process is the unique strong solution of the Langevin equation driven by the fractional Brownian motion B^H

$$X_0 = 0, \qquad dX_t = a(b - X_t)dt + \sigma d B_t^H, \qquad t \geq 0, \tag{3}$$

Here $a > 0, b \in \mathbb{R}$ are real drift parameters and $\sigma > 0$ is the volatility of the model.

- When $b = 0$, X is a fractional Ornstein-Uhlenbeck process
- The fractional Vasicek model has received a lot of attention, because of its potential for modelling purpose and since one can use the powerful toolbox of Gaussian analysis to deal with it.

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Motivation

- But in some situation, the Gaussian assumption may be implausible. For example in hydrology when we would like to analyze river-flow time series.

- This is why we propose to look at a non-Gaussian extension:

$$dX_t = a(b - X_t)dt + \sigma dZ_t^{q,H}, \qquad X_0 = 0.$$

Here $Z^{q,H}$ stands for a Hermite process.

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Vasicek-type model driven by Hermite processes

The (non-stationary) Vasicek-type model driven by Hermite processes is the unique (pathwise) solution of the Langevin equation driven by Hermite process $Z^{q,H}$

$$dX_t = a(b - X_t)dt + \sigma dZ_t^{q,H}, \qquad X_0 = 0. \tag{4}$$

It is easily shown that the solution of (4) is given by

$$X_t = b(1 - e^{-at}) + \sigma \int_0^t e^{-a(t-u)} dZ_u^{q,H}.$$

The choice $q = 1$ in (4) corresponds to fractional Vasicek model. When $b = 0$, one gets Hermite Ornstein-Uhlenbeck process.

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Problem we have looked at:

Recall (4): $dX_t = a(b - X_t)dt + \sigma dZ_t^{q,H}$.
Here $H \in (\frac{1}{2}, 1)$, σ are known (assume $\sigma = 1$), a and b are unknown.

- Construct estimators for a and b.
- Study their asymptotic properties (their consistency as well as their fluctuations around the true value of the parameter) based on a continuous-time observation of X.
- Do our estimators for a and b have the same asymptotic behavior when $q = 1$ (fBm case, fractional model (3)) and $q \geq 2$ (non-Gaussian case, model (4))?

  Answering this question is equivalent to understand whether the Gaussian feature of the fBm really matters when estimating the unknown parameters in the fractional Vasicek model (3) .

Estimators for drift parameters

Recall (4): $dX_t = a(b - X_t)dt + dZ_t^{q,H}, \qquad X_0 = 0.$

Strong solution:

$$X_t = b(1 - e^{-at}) + \int_0^t e^{-a(t-s)} dZ_s^{q,H}.$$

Assume: $q \geq 1, H \in (\frac{1}{2}, 1)$ are known; $a > 0$ and $b \in \mathbb{R}$ are unknown.

Definition

We define estimators for drift parameters a and b in (4) as follows:

$$\widehat{a}_T = \left(\frac{\alpha_T}{H\Gamma(2H)}\right)^{-\frac{1}{2H}}, \text{ where } \alpha_T = \frac{1}{T}\int_0^T X_t^2 dt - \left(\frac{1}{T}\int_0^T X_t dt\right)^2,$$

$$\widehat{b}_T = \frac{1}{T}\int_0^T X_t dt. \qquad (5)$$

Strong consistency of the estimators

Theorem

For any $q \geq 1$ and any $H \in (\frac{1}{2}, 1)$, we have, as $T \to \infty$

$$(\widehat{a}_T, \widehat{b}_T) \stackrel{\text{a.s.}}{\to} (a, b).$$

The proof of this theorem relies on the following proposition.

Proposition

For all $q \geq 1$ and $H \in (\frac{1}{2}, 1)$, one has, as $T \to \infty$

$$\frac{1}{T}\int_0^T X_t dt \stackrel{\text{a.s.}}{\to} b \tag{6}$$

$$\frac{1}{T}\int_0^T X_t^2 dt \stackrel{\text{a.s.}}{\to} b^2 + a^{-2H} H\Gamma(2H). \tag{7}$$

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$$\frac{1}{T}\int_0^T X_t^2 dt \overset{\text{a.s.}}{\to} b^2 + a^{-2H} H\Gamma(2H). \tag{7}$$

Key steps of the proof of strong consistency

$X_t = h(t) + Y_t$ where $h(t) = b(1 - e^{-at})$, $Y_t = \int_0^t e^{-a(t-u)} dZ_u^{q,H}$.

Proof (6): We write

$$\frac{1}{T}\int_0^T X_t dt = \frac{b}{T}\int_0^T (1 - e^{-at})dt + \frac{1}{T}\int_0^T Y_t dt.$$

- Check that $\frac{b}{T}\int_0^T (1 - e^{-at})dt \to b$ (Lebesgue dominated convergence)
- Prove that $\frac{1}{T}\int_0^T Y_t dt \to 0$ almost surely
  Since $\frac{1}{T}\int_0^T Y_t dt$ belongs to the qth Wiener chaos, it enjoys the hypercontractivity property + using Borel-Cantelli lemma to obtain desired conclusion.

Proof (7):

$$\frac{1}{T}\int_0^T X_t^2 dt = \frac{1}{T}\int_0^T h(t)^2 dt + \frac{2}{T}\int_0^T h(t) Y_t dt + \frac{1}{T}\int_0^T Y_t^2 dt.$$

1. Check $\frac{1}{T}\int_0^T h(t)^2 dt \to b^2$.
2. Prove $\frac{1}{T}\int_0^T h(t) Y_t dt \overset{\text{a.s}}{\to} 0$. Hint: $T^{-H}\int_0^T h(t) Y_t dt \overset{\text{law}}{\to} \frac{b}{a} Z_1^{q,H}$
3. Show $\frac{1}{T}\int_0^T Y_t^2 dt \overset{\text{a.s}}{\to} a^{-2H} H\Gamma(2H)$.
   Hint: First, observe that

$$\frac{1}{T}\int_0^T E[Y_t^2] dt \to a^{-2H} H\Gamma(2H)$$

   Secondly, applying the next following results on quadratic functional of long memory moving average process $(Y_t)_{t\geq 0}$

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Proof (7) Recall $Y_t = \int_0^t e^{-a(t-u)} dZ_u^{q,H}$. As $T \to \infty$, we have

- If $q \geq 2$ or ($q = 1$ and $H > \frac{3}{4}$):

$$T^{\frac{2}{q}(1-H)-1} \int_0^T (Y_t^2 - E[Y_t^2])dt \overset{\text{law}}{\to} \text{Rosenblatt r.v}$$

- If $q = 1$ and $H \in (\frac{1}{2}, \frac{3}{4})$:

$$T^{-\frac{1}{2}} \int_0^T (Y_t^2 - E[Y_t^2])dt \overset{\text{law}}{\to} \text{Standard Gaussian r.v}$$

- If $q = 1$ and $H = \frac{3}{4}$

$$(T \log(T))^{-\frac{1}{2}} \int_0^T (Y_t^2 - E[Y_t^2])dt \overset{\text{law}}{\to} \text{Standard Gaussian r.v}$$

The asymptotic behaviour holds for general long memory moving average processes $Y_t = \int_0^t x(t-u) dZ_u^{q,H}$ satisfying that

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}_+} |x(u)x(v)||u-v|^{2H-2} dudv < \infty.$$

Fluctuations of the estimators: non-Gaussian case

Theorem

Let $X = (X_t)_{t\geq 0}$ *be the unique strong solution to* $dX_t = a(b - X_t)dt + dZ_t^{q,H}$ *with* $a > 0, b \in \mathbb{R}$.

For any $q \geq 2$ *and any* $H \in (\frac{1}{2}, 1)$*:*

$$\left(T^{\frac{2}{q}(1-H)}\{\widehat{a}_T - a\}, T^{1-H}\left\{\widehat{b}_T - b\right\}\right) \overset{\text{law}}{\to} \left(-\frac{a^{1-\frac{2}{q}(1-H)}}{2H^2\Gamma(2H)} G_\infty, \frac{1}{a} Z_1^{q,H}\right),$$

as $T \to \infty$*. Here* G_∞ *is distributed according to Rosenblatt distribution of parameter* $1 + (2H-2)/q$ *up to an explicit constant.* $Z_1^{q,H}$ *is a Hermite random variable (Hermite process evaluated at* 1*).*

Fluctuations of the estimators: Gaussian case

Theorem

- *Case $q = 1$ and $H < \frac{3}{4}$:*

$$\left(\sqrt{T}\{\widehat{a}_T - a\}, T^{1-H}\{\widehat{b}_T - b\}\right) \overset{\text{law}}{\to} \left(-\frac{a^{1+4H}\sigma_H}{2H^2\Gamma(2H)} N, \frac{1}{a}N'\right).$$

- *Case $q = 1$ and $H = \frac{3}{4}$:*

$$\left(\sqrt{\frac{T}{\log T}}\{\widehat{a}_T - a\}, T^{\frac{1}{4}}\left\{\widehat{b}_T - b\right\}\right) \overset{\text{law}}{\to} \left(\frac{3}{4}\sqrt{\frac{a}{\pi}} N, \frac{1}{a}N'\right).$$

- *Case $q = 1$ and $H > \frac{3}{4}$:*

$$\left(T^{2-2H}\{\widehat{a}_T - a\}, T^{1-H}\left\{\widehat{b}_T - b\right\}\right) \overset{\text{law}}{\to} \left(\frac{-a^{2H-1}}{2H^2\Gamma(2H)}\left(G_\infty - (B_1^H)^2\right), \frac{1}{a}B_1^H\right)$$

Keys of proof of fluctuations

Proposition (Definition)

Assume either ($q = 1$ and $H > \frac{3}{4}$) or $q \geq 2$. Fix $T > 0$. Let $U_T = (U_T(t))_{t\geq 0}$ be a process defined as $U_T(t) = \int_0^t e^{-T(t-u)} dZ_u^{q,H}$. Finally, let G_T be the random variable defined as

$$G_T = T^{\frac{2}{q}(1-H)+2H} \int_0^1 \left(U_T(t)^2 - E[U_T(t)^2]\right) dt.$$

Then G_T converges in $L^2(\Omega)$ to a limit written G_∞. Moreover, $G_\infty / B_{H,q}$ is distributed according to the Rosenblatt distribution of parameter $1 - \frac{2}{q}(1-H)$, where $B_{H,q}$ is an explicit cst.

Keys of proof of fluctuations

Fluctuations of $\widehat{b}_T$

$$T^{1-H}\{\widehat{b}_T - b\} = T^{1-H}\left\{\frac{1}{T}\int_0^T Y_t dt - \frac{b}{T}\int_0^T e^{-at} dt\right\}$$
$$= \frac{Z_T^{q,H}}{aT^H} + O(T^{-H})$$

Fluctuations of $\widehat{a}_T$

$$T^{\frac{2}{q}(1-H)}\{\widehat{a}_T - a\}$$
$$= -\frac{a^{1+2H}}{2H^2\Gamma(2H)}\left(T^{\frac{2}{q}(1-H)}A_T - T^{\frac{2}{q}(1-H)-2}\frac{(Z_T^{q,H})^2}{a^2}\right) + o(1),$$

Where $A_T = \frac{1}{T}\int_0^T (Y_t^2 - E[Y_t^2])dt$

Keys of proof of fluctuations

Fluctuations of $(\widehat{a}_T, \widehat{b}_T)$

We consider first the case ($q = 1$ and $H > \frac{3}{4}$) or ($q \geq 2$). Since $Z^{q,H}$ satisfies the scaling property, we can write

$$\left(T^{\frac{2}{q}(1-H)}A_T,\, T^{-H}Z_T^{q,H}\right) \overset{\text{law}}{=} \left(a^{-\frac{2}{q}(1-H)-2H}\, G_{aT},\, Z_1^{q,H}\right),$$

and we deduce from the L^2-convergence of G_T that

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Conclusion

Question: Do our estimators for a and b have the same asymptotic behavior in case of Gaussian and non-Gaussian ($q \geq 2$) or not ?

Answer:

- The strong consistency of $(\widehat{a}_T, \widehat{b}_T)$ is universal for any Vasicek type model driven by Hermite process as a noise, no matter that it is Gaussian or not.

- The fluctuations of our estimators around the true value of the drift parameters depend heavily on the order q and Hurst parameter H of the underlying Hermite process.

This gives us some hints to understand how much the fractional Vasicek model relies on the Gaussian feature.

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