Testing equality of spectral density operators for functional linear processes

Anne Leucht
Otto-Friedrich-Universität Bamberg

joint work with
E. Paparoditis (U Cyprus), D. Rademacher (U Heidelberg), T. Sapatinas (U Cyprus)

Ecodep Seminary

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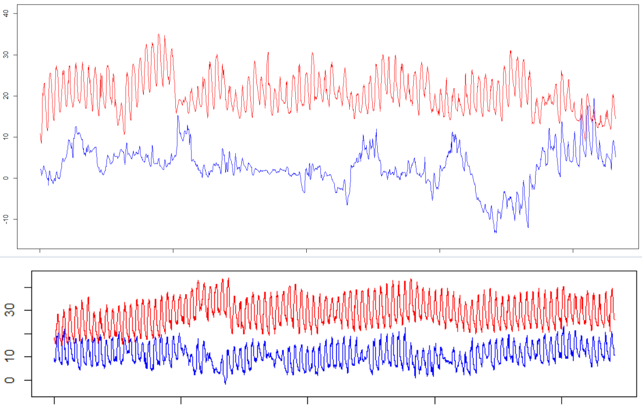
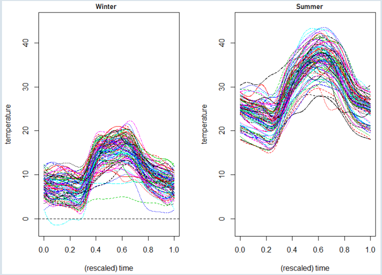
1 Motivating Example

Analysis of temperature data - comparison of seasons

- **top:** temperature data Dresden (Germany) 2020/21
  90 summer days, 90 winter days
- **bottom:** temperature data Nicosia (Cyprus) 2006/07
  96 summer days, 96 winter days

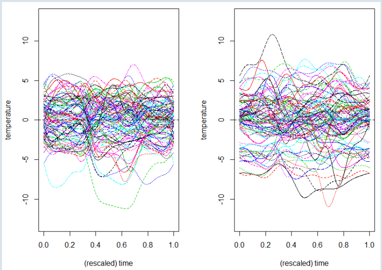


1 Motivating Example



- daily temperature curves in Nicosia: (measurements every 15 minutes)
  - left: winter (01.12.2006-02.03.2007)
  - right: summer (01.06.2007-31.08.2007)

$\rightarrow$ winter & summer differ in intra-day pattern



$\rightarrow$ centering around averages

? difference in variability of summer and winter

1 Motivating Example

- framework: **functional time series**
  - X_t centered temperature curve at day t in winter
  - Y_t centered temperature curve at day t in summer

Assumption (A1)

(i) $(X_t)_t$ and $(Y_t)_t$ are independent functional linear processes,

$$X_t = \sum_{j\in\mathbb{Z}} A_j(\varepsilon_{t-j}) \quad \text{and} \quad Y_t = \sum_{j\in\mathbb{Z}} B_j(e_{t-j}), \quad t \in \mathbb{Z},$$

with values in $L^2_{\mathbb{R}}([0,1],\mu)$

(ii) $(\varepsilon_t)_{t\in\mathbb{Z}}$ and $(e_t)_{t\in\mathbb{Z}}$: two independent i.i.d. mean zero Gaussian processes

(iii) $(A_j)_{j\in\mathbb{Z}}$ and $(B_j)_{j\in\mathbb{Z}}$ bounded linear operators with $A_0 = B_0$ being identity operator and, satisfy $\sum_{j\in\mathbb{Z}} |j| \left(\|A_j\|_{\mathcal{L}} + \|B_j\|_{\mathcal{L}}\right) < \infty$
($\|\cdot\|_{\mathcal{L}}$ operator norm)

- next: mathematical formulation of variability

1 Motivating Example

$\rightarrow$ using autocovariance operators $\mathcal{R}_{X,h}$ induced by right-integration of

$$r_h^X\colon [0,1]\times[0,1]\to\mathbb{R}, \qquad r_h^X(\tau_1,\tau_2)=cov(X_h(\tau_1),X_0(\tau_2)),$$

that is

$$\mathcal{R}_{X,h}(v(\tau_1))=\int_{[0,1]} r_h^X(\tau_1,\tau_2)\,v(\tau_2)\,d\tau_2, \quad v\in L^2_{\mathbb{R}}([0,1],\mu)$$

- **test problem:**

$$\mathcal{H}_0\colon \mathcal{R}_{X,h}=\mathcal{R}_{Y,h} \quad \forall h \quad \text{against} \quad \mathcal{H}_1\colon \exists\, h\in\mathbb{N}_0:\ \mathcal{R}_{X,h}\neq\mathcal{R}_{Y,h}$$

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2 Some spectral theory for functional time series

- recall the test problem

$$\mathcal{H}_0\colon \mathcal{R}_{X,h} = \mathcal{R}_{Y,h} \quad \forall h \quad \text{against} \quad \mathcal{H}_1\colon \exists\, h \in \mathbb{N}_0 : \ \mathcal{R}_{X,h} \neq \mathcal{R}_{Y,h}$$

- expectation for many problems: if null does not hold, then deviation of autocovariance operators in many lags

$\rightarrow$ hard to interpret

- **idea: spectral approach**
  - autocovariance (kernel): superposition of periodic functions with different frequencies λ and different magnitudes
  - if $\mathcal{H}_0$ is rejected, then further analysis if main difference is due to large or small frequencies (equiv. short or long periods)

- well-known from univariate times series analysis: benefit from one-to-one correspondence between second order structure and spectral density (also from mathematical perspective)

2 Some spectral theory for functional time series

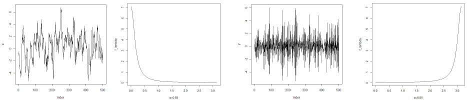
- if autocovariances $(r_h^Z)_h$ of a univariate time series $(Z_t)_t$ are absolutely summable then the **spectral density** is defined as

$$f_Z(\lambda) := \frac{1}{2\pi} \sum_{h \in \mathbb{Z}} r_h^Z \, e^{-ih\lambda}, \quad \lambda \in (-\pi, \pi]$$

- and **inversion formula** holds

$$r_h^Z = \int_{-\pi}^{\pi} e^{ih\lambda} \, f_Z(\lambda) \, d\lambda, \quad h \in \mathbb{Z}$$

$\rightarrow$ equality of second order structure equivalent to equality of spectral densities



! spectral approach to corresponding test problem for multivariate time series successfully applied e.g. by Eichler (2008), Dette & Paparoditis (2009)

$\rightarrow$ **in this talk:** generalization to functional linear processes

2 Some spectral theory for functional time series

Definition 2.1 (Spectral density kernels & $\sim$ operators).

For a functional linear process satisfying (A1) the **spectral density kernel** at frequency $\lambda \in (-\pi, \pi]$ is given by

$$\mathbf{f}_{\mathbf{X},\lambda}(\sigma,\tau) = \frac{\mathbf{1}}{\mathbf{2}\pi} \sum_{\mathbf{h}\in\mathbb{Z}} \mathbf{e}^{-\mathbf{i}\lambda\mathbf{h}} \mathbf{r}_{\mathbf{X},\mathbf{h}}(\sigma,\tau), \quad \sigma, \tau \in [0,1].$$

The operator $\mathcal{F}_{X,\lambda}$ induced by right-integration is called **spectral density operator**.

- ✓ $f_{X,\lambda}$ converges absolutely in L^2, the inversion formula holds
- ✓ $\mathcal{F}_{X,\lambda}$ is a self-adjoint, nonnegative definite operator
- ! overview in more general context: see e.g. Panaretos & Tavakoli (2013)
- **test problem:** spectral reformulation

$$\mathcal{H}_0\colon\ \mathcal{F}_{X,\lambda} = \mathcal{F}_{Y,\lambda} \quad \text{for } \mu\text{-almost all } \lambda \in (-\pi,\pi], \quad \text{versus}$$
$$\mathcal{H}_1\colon\ \mathcal{F}_{X,\lambda} \neq \mathcal{F}_{Y,\lambda} \quad \forall \lambda \in A \text{ for some } A \subset [0,\pi] \text{ with } \mu(A) > 0.$$

2 Some spectral theory for functional time series

- **1st step:** generalization of periodogram from multivariate time series to **periodogram kernel**

$$\widehat{p}_{X,\lambda}(\sigma,\tau) = \frac{1}{2\pi T} \sum_{s_1,s_2=1}^{T} X_{s_1}(\sigma)\, X_{s_2}(\tau)\, e^{-i\lambda(s_1-s_2)}, \quad \sigma,\tau \in [0,1],$$

- well-known from multivariate time series: periodogram not consistent
- **2nd step:** kernel-type smoothing of periodogram: $\lambda_t = 2\pi \frac{t}{T}$, $N = \left\lfloor \frac{T-1}{2} \right\rfloor$
spectral density estimator

$$\hat{f}_{X,\lambda}(\sigma,\tau) = \frac{1}{bT} \sum_{t=-N}^{N} W\left(\frac{\lambda-\lambda_t}{b}\right) \widehat{p}_{X,\lambda_t}(\sigma,\tau), \quad \sigma,\tau \in [0,1],$$

- **3rd step:** **estimated spectral density operator**

$$\widehat{\mathcal{F}}_{X,\lambda}(v(\sigma)) = \int_{(-\pi,\pi]} \hat{f}_{X,\lambda}(\sigma,\tau)\, v(\tau)\, d\tau, \quad v \in L^2_{\mathbb{R}}([0,1],\mu)$$

- (non-)asymptotic properties: IMSE, CLT,...
e.g. in Panaretos & Tavakoli (2013), Cerovecki, Hörmann (2017)

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3 Test statistic and its asymptotics

- recall **test problem:**

$$\mathcal{H}_0\colon\ \mathcal{F}_{X,\lambda} = \mathcal{F}_{Y,\lambda} \quad \text{for } \mu\text{-almost all } \lambda \in (-\pi,\pi], \quad \text{versus}$$
$$\mathcal{H}_1\colon\ \mathcal{F}_{X,\lambda} \neq \mathcal{F}_{Y,\lambda} \quad \forall \lambda \in A \text{ for some } A \subset [0,\pi] \text{ with } \mu(A) > 0.$$

- projection-based approach for fixed frequencies (+ multiple testing): Tavakoli & Panaretos (2016)

- here: evaluation of all frequencies

→ **test statistic:**

$$\mathcal{U}_T = \int_{(-\pi,\pi]} \|\widehat{\mathcal{F}}_{X,\lambda} - \widehat{\mathcal{F}}_{Y,\lambda}\|_{HS}^2 \, d\lambda$$
$$= \int_{(-\pi,\pi]} \iint_{[0,1]^2} |\hat{f}_{X,\lambda}(\sigma,\tau) - \hat{f}_{Y,\lambda}(\sigma,\tau)|^2 \, d\sigma \, d\tau \, d\lambda$$

- related work for corresponding "relevant hypotheses": van Delft & Dette (2020)

3 Test problem and asymptotics of the test statistic

Theorem 3.1 (Asymptotics under $\mathcal{H}_0$ and $\mathcal{H}_1$).

Assume that (A1) holds and that

(i) $b \sim T^{-\nu}$ for some $\nu \in (1/4, 1/2)$,

(ii) W is bounded, symmetric, positive, Lipschitz continuous, has bounded support on $(-\pi, \pi]$ and satisfies $\int_{-\pi}^{\pi} W(x)\, dx = 2\pi$.

Then, **under $\mathcal{H}_0$**,

$$\sqrt{\mathbf{b}}\mathbf{T}\,\mathcal{U}_{\mathbf{T}} - \mathbf{b}^{-1/2}\mu_0 \xrightarrow{\mathbf{d}} \mathbf{Z} \sim \mathcal{N}(\mathbf{0}, \theta_0^2),$$

where

$$\mu_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} \{\mathrm{trace}(\mathcal{F}_{X,\lambda})\}^2 \, d\lambda \int_{-\pi}^{\pi} W^2(u)\, du$$

$$\theta_0^2 = \frac{4}{\pi^2} \int_{-2\pi}^{2\pi} \left\{ \int_{-\pi}^{\pi} W(u) W(u-x)\, du \right\}^2 dx \int_{-\pi}^{\pi} \|\mathcal{F}_{X,\lambda}\|_{HS}^4 \, d\lambda$$

and, **under $\mathcal{H}_1$**,

$$\sqrt{\mathbf{b}}\mathbf{T}\,\mathcal{U}_{\mathbf{T}} - \mathbf{b}^{-1/2}\mu_0 \xrightarrow{\mathbf{P}} \infty.$$

3 Test problem and asymptotics of the test statistic

- **Comments on assumptions**
  - ! Assumptions (i) and (ii) equivalent to assumptions for multivariate time series in Dette and Paparoditis (2009)
  - main benefit from using linear processes:
    - periodogram operator of $(X_t)_t$ can be easily traced back to periodogram operator of $(\varepsilon_t)_t$
    - Gaussianity only needed to prove normality, but not to derive μ_0, θ_0^2

→ **consistent asymptotic α-test**: reject $\mathcal{H}_0$ if

$$\mathcal{T}_{\mathcal{U}} = \frac{\sqrt{b}T\,\mathcal{U}_T - b^{-1/2}\widehat{\mu}_0}{\widehat{\theta}_0} \geq z_{1-\alpha},$$

where

- $z_{1-\alpha}$ is the upper $1-\alpha$ quantile of $\mathcal{N}(0,1)$
- $\widehat{\mu}_0$ and $\widehat{\theta}_0$ are consistent estimators of μ_0 and θ_0
  (e.g. obtained by substitution of the unknown spectral density kernel $f_{X,\lambda}$ by **pooled estimator** $\widehat{f}_\lambda(\tau,\sigma) = \widehat{f}_{X,\lambda}(\tau,\sigma)/2 + \widehat{f}_{Y,\lambda}(\tau,\sigma)/2$)

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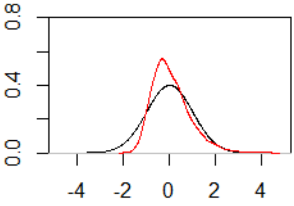
4 Bootstrapping the test statistic

- known from finite-dimensional case:
  - convergence to normality is very slow
  - bootstrap approaches can improve performance of tests



T=500, b=0.04

- 2 main issues:
  - complicated dependence structure
  - infinite dimensionality
- towards a solution...
  - consider building blocks of periodogram

$$J_{X,\lambda} = \Big(J_{X,\lambda}(s_j) = (2\pi T)^{-1/2} \sum_{t=1}^{T} X_t(s_j) e^{-it\lambda},\ j = 1, 2, \ldots, k \Big),$$

  - Cerovecki, Hörmann (2017):
    - ⋆ $J_{X,\lambda} \xrightarrow{d} J \sim \mathcal{N}_C(0, \Sigma_\lambda)$ with $\Sigma_\lambda = (f_{X,\lambda}(s_{j_1}, s_{j_2}))_{j_1, j_2}$
    - ⋆ J_{X,λ_1} and J_{X,λ_2} are asymptotically independent for $0 < \lambda_1 < \lambda_2 < \pi$

4 Bootstrapping the test statistic

Algorithm

1. Generate $J^*_{X,0} = J^*_{Y,0} = 0$ and independent vectors

$$J^*_{X,\lambda_t} \sim \mathcal{N}_C(0, \widehat{\Sigma}_{\lambda_t}) \quad \text{and} \quad J^*_{Y,\lambda_t} \sim \mathcal{N}_C(0, \widehat{\Sigma}_{\lambda_t}),$$

independently for $\lambda_1, \ldots, \lambda_N$, where $\widehat{\Sigma}_\lambda = (\hat{f}_\lambda(s_{j_1}, s_{j_2}))_{j_1,j_2}$ with $\hat{f}_\lambda = \frac{1}{2}\hat{f}_{X,\lambda} + \frac{1}{2}\hat{f}_{Y,\lambda}$.

2. For $\sigma, \tau \in \{s_1, s_2, \ldots, s_k\}$ and $t = 1, \ldots, N$, calculate

$$p^*_{X,\lambda_t}(\sigma,\tau) = J^*_{X,\lambda_t}(\sigma)\,\overline{J}^*_{X,\lambda_t}(\tau) \quad \text{and} \quad p^*_{Y,\lambda_t}(\sigma,\tau) = J^*_{Y,\lambda_t}(\sigma)\,\overline{J}^*_{Y,\lambda_t}(\tau)$$

while, for $t = -1, -2, \ldots, -N$, set

$$p^*_{X,\lambda_t}(\sigma,\tau) = \overline{p}^*_{X,-\lambda_t}(\sigma,\tau) \quad \text{and} \quad p^*_{Y,\lambda_t}(\sigma,\tau) = \overline{p}^*_{Y,-\lambda_t}(\sigma,\tau).$$

4 Bootstrapping the test statistic

Algorithm (cont'd)

3. For $\sigma, \tau \in \{s_1, s_2, \dots, s_k\}$, let

$$\hat{f}^*_{X,\lambda_t}(\sigma,\tau) = \frac{1}{bT} \sum_{s=-N}^{N} W\left(\frac{\lambda_t - \lambda_s}{b}\right) p^*_{X,\lambda_s}(\sigma,\tau)$$

and

$$\hat{f}^*_{Y,\lambda_t}(\sigma,\tau) = \frac{1}{bT} \sum_{s=-N}^{N} W\left(\frac{\lambda_t - \lambda_s}{b}\right) p^*_{Y,\lambda_s}(\sigma,\tau).$$

4. Calculate bootstrap test statistic $\mathcal{U}^*_{T,k}$ given by

$$\mathcal{U}^*_{T,k} = \frac{2\pi}{Tk^2} \sum_{l=-N}^{N} \sum_{i,j=1}^{k} \left| \widehat{f}^*_{X,\lambda_l}(s_i,s_j) - \widehat{f}^*_{Y,\lambda_l}(s_i,s_j) \right|^2$$

and $$\mathcal{T}^*_{\mathcal{U},k} = (\sqrt{b}T\,\mathcal{U}^*_{T,k} - b^{-1/2}\widehat{\mu}^*_0)/\widehat{\theta}^*_0.$$

5. Reject $\mathcal{H}_0$ if $\mathcal{T}_{\mathcal{U}} > t^*_{1-\alpha}$ with $t^*_{1-\alpha}$ denoting the $(1-\alpha)$ quantile of $\mathcal{T}^*_{\mathcal{U},k}$.

4 Bootstrapping the test statistic

Brief comment on discretization

- in practice: typically X_t and Y_t are observed only at finitely many sampling points (transformation to functional objects using basis functions in L^2)

→ natural choice of $0 \leq s_1 < s_2 < \cdots < s_k \leq 1$: sampling points of X_t and Y_t

- transformation of J^*-variables to functional objects possible

→ bootstrap approximation of the test statistic $\mathcal{U}_T$:

$$\mathcal{U}_T^* = \frac{2\pi}{T} \sum_{l=-N}^{N} \int_0^1 \int_0^1 \left| \widehat{f}^*_{X,\lambda_l}(\tau,\sigma) - \widehat{f}^*_{Y,\lambda_l}(\tau,\sigma) \right|^2 d\tau d\sigma$$

- $\mathcal{U}_{T,k}^*$ and $\mathcal{U}_T^*$ will lead to the same result, provided that $k \to \infty$ as $T \to \infty$

4 Bootstrapping the test statistic

Theorem 4.1 (Bootstrap validity).

Suppose that prerequisites of Theorem 3.1 are satisfied. Then, conditional on $X_1, \ldots, X_T, Y_1, \ldots, Y_T$, as $T \to \infty$,

$$\sqrt{\mathbf{b}}\mathbf{T}\,\mathcal{U}^*_{\mathbf{T}} - \mathbf{b}^{-1/2}\widetilde{\mu}_{\mathbf{0}} \xrightarrow{\mathbf{d}} \widetilde{\mathbf{Z}} \sim \mathcal{N}(\mathbf{0}, \widetilde{\theta}^{\mathbf{2}}_{\mathbf{0}}),$$

in probability, where

$$\widetilde{\mu}_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} \{\mathrm{trace}(\mathcal{F}_{X,\lambda}/2 + \mathcal{F}_{Y,\lambda}/2)\}^2 d\lambda \int_{-\pi}^{\pi} W^2(u)\, du,$$

$$\widetilde{\theta}^2_0 = \frac{4}{\pi^2} \int_{-2\pi}^{2\pi} \left\{\int_{-\pi}^{\pi} W(u)W(u-x)\, du\right\}^2 dx \int_{-\pi}^{\pi} \|\mathcal{F}_{X,\lambda}/2 + \mathcal{F}_{Y,\lambda}/2\|^4_{HS}\, d\lambda.$$

! on bootstrap side (A1) can be relaxed to

$$\sup_{\lambda_t \in \{2\pi k/T | k=1,\ldots,N\}} \left| \int_0^1 \int_0^1 \left(\widehat{f}_{\lambda_t}(\sigma,\tau) - f_{\lambda_t}(\sigma,\tau)\right) d\sigma\, d\tau \right| = o_P(\sqrt{b})$$

with $f_\lambda = 0.5\, f_{X,\lambda} + 0.5\, f_{Y,\lambda}$

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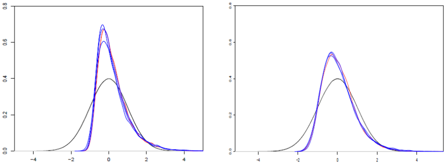
5 Numerical examples & conclusion

Simulations

- functional MA processes with Brownian bridge innovations

$$X_t = A_1(\varepsilon_{t-1}) + a_2\,\varepsilon_{t-2} + \varepsilon_t, \qquad Y_t = A_1(e_{t-1}) + e_t,$$

$a_2 \in [0,1)$, A_1 kernel operator with $a_1(u,v) = \frac{e^{-(u^2+v^2)/2}}{4\int_0^1 e^{-t^2}dt}$,



- left: T=50
- right: T=500

	T=50	b=0.2		T=100	b=0.2	
a_2	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.10$	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.10$
0.0	0.010	0.048	0.096	0.008	0.046	0.080
0.2	0.016	0.082	0.158	0.028	0.112	0.196
0.6	0.178	0.390	0.518	0.374	0.622	0.766
1.0	0.488	0.768	0.872	0.874	0.966	0.990

Table: Empirical size and power of the bootstrap studentized test.

5 Numerical examples & conclusion

Choice of the bandwidth

- adapting approach of Robinson (1991) for multivariate time series
- define averaged (pooled) periodogram

$$\widehat{I}_T(\lambda) = \frac{1}{k^2} \sum_{r=1}^{k} \sum_{s=1}^{k} \left\{ \frac{1}{2} \widehat{p}_{X,\lambda}(\sigma_r, \tau_s) + \frac{1}{2} \widehat{p}_{Y,\lambda}(\sigma_r, \tau_s) \right\}$$

→ periodogram at frequency λ of the pooled, real-valued univariate process

$$\left\{ V_t = \frac{1}{2} \int_0^1 X_t(s) ds + \frac{1}{2} \int_0^1 Y_t(s) ds,\ t \in \mathbb{Z} \right\}$$

→ averaged pooled spectral density estimator of $\{V_t, t \in \mathbb{Z}\}$

$$\widehat{g}^{(b)}(\lambda_t) = \frac{1}{Tb} \sum_{s=-N}^{N} W\left(\frac{\lambda_t - \lambda_s}{b} \right) \widehat{I}_T(\lambda_s)$$

5 Numerical examples & conclusion

Choice of the bandwidth

- **cross validation**: $\widehat{b}_{CV} = \arg\min CV(b)$ with

$$CV(b) = \frac{1}{N} \sum_{t=1}^{N} \left\{ \log(\widehat{g}_{-t}^{(b)}(\lambda_t)) + \frac{\widehat{I}_T(\lambda_t)}{\widehat{g}_{-t}^{(b)}(\lambda_t)} \right\}$$

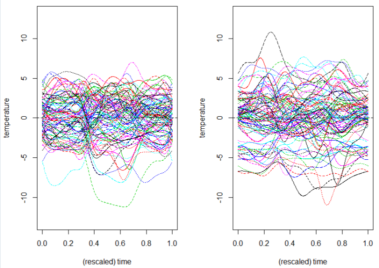
  over a grid of values of b

- with leave-one-out kernel estimator of $g(\lambda)$

$$\widehat{g}_{-t}^{(b)}(\lambda_t) = \frac{1}{Tb} \sum_{\substack{s=-N \\ s \neq \pm t}}^{N} W\left(\frac{\lambda_t - \lambda_s}{b}\right) \widehat{I}_T(\lambda_s)$$

5 Numerical examples & conclusion

Data Example



- $\mathcal{H}_0: \ \mathcal{F}_{\text{winter},\lambda} = \mathcal{F}_{\text{summer},\lambda}$ for $\mu-$almost all $\lambda \in (-\pi, \pi]$,
- bandwidth obtained by CV

$\rightarrow$ Nicosia: *p*-value 0.03

$\rightarrow$ rejection of $\mathcal{H}_0$ at most of the classical levels

$\rightarrow$ Dresden: *p*-value 0.0011

$\rightarrow$ rejection of $\mathcal{H}_0$ at essentially all classical levels

- further analysis: identification of main contributions to test statistic in terms of frequencies and time of day

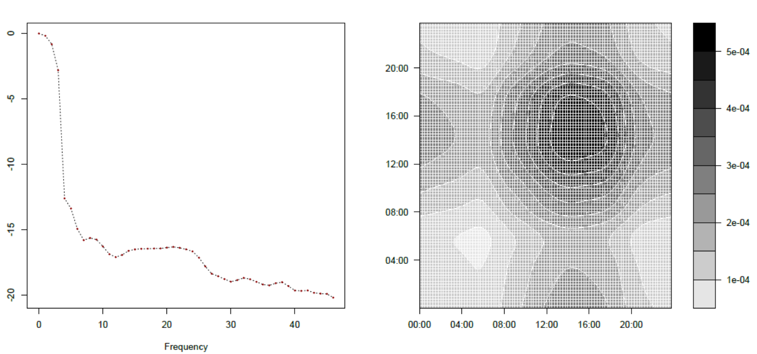
$\rightarrow$ split up test statistic accordingly

5 Numerical examples & conclusion

Data Example

Further analysis for Nicosia

$$Q_{T,\lambda_l} = 2\pi\sqrt{b}\|\widehat{\mathcal{F}}^*_{X,\lambda_l} - \widehat{\mathcal{F}}^*_{Y,\lambda_l}\|^2_{HS}/\widehat{\theta}_0, \quad D_T(\sigma,\tau) = \frac{2\pi}{T}\sum_{l=-N}^{N}\left|\widehat{f}^*_{X,\lambda_l}(\sigma,\tau) - \widehat{f}^*_{Y,\lambda_l}(\sigma,\tau)\right|^2$$



(log-scale)

5 Numerical examples & conclusion

Conclusion

✓ test for whole second order structure of independent functional linear processes in frequency domain

! bootstrap performs better than asymptotics alone

! (A1) can be strongly relaxed when proving

- consistency of the test
- bootstrap validity

- composition of the test statistic allows further analysis w.r.t. specific frequencies and certain time points (within a day) in applications
- propose a bandwidth selection algorithm
- extension to two samples with different sample size possible

Thank you for your attention!

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