HJB equation on process space
Application to mean field control

Nizar Touzi
NYU Tandon School of Engineering

Joint work with Jianjun Zhou & Jianfeng Zhang

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HJB equation and Viscosity solutions

HJB equation in $\mathbb{R}^d$:

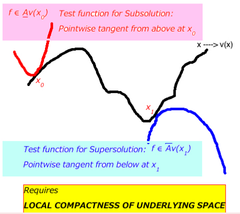
$$\partial_t u(t,x) + H\big(t,x,u(t,x),Du(t,x),D^2u(t,x)\big) = 0,\ \ t<T,\ \ x\in\mathbb{R}^d$$

$$u\big|_{t=T} = g,\ \ x\in\mathbb{R}^d$$

where $H(\cdots) := \inf_{a\in A}\Big\{b(.,a)\cdot Du + \frac{1}{2}\sigma\sigma^{\intercal}(.,a):D^2u - k(.,a)u + f(.,a)\Big\}$

$$\overline{\mathcal{A}}u(t,x) := \Big\{\varphi\in C^2 : (\varphi-u_*)(t,x) = \max(\varphi-u_*)\Big\}$$

$$\underline{\mathcal{A}}u(t,x) := \Big\{\varphi\in C^2 : (\varphi-u^*)(t,x) = \min(\varphi-u^*)\Big\}$$



Consistency : For $u\in C^{1,2}$, u classical sol. **iff** u viscosity sol.
Stability : v_ε be visco supersol, loc bdd in (ε,t,m). Then

$$\underline{v}(t,x) := \liminf_{(\varepsilon,t',x')\to(0,t,x)} v_\varepsilon(t',x') \text{ is a visco supersol}$$

Existence : representation as value function of a control problem

HJB equation and stochastic control

HJB equation in $\mathbb{R}^d$:

$$\partial_t u + \inf_{a \in A} \left\{ b(.,a) \cdot Du + \frac{1}{2}\sigma\sigma^\intercal(.,a) : D^2 u - k(.,a)u + f(.,a) \right\} = 0$$

$$u\big|_{t=T} = g$$

Control process α prog. meas. with values in A

Controlled state process driven by BM W in $\mathbb{R}^d$:

$$X_t^{t,x} = x \text{ and } dX_s^{t,x} = b(s, X_s^{t,x}, \alpha_s)ds + \sigma(s, X_s^{t,x}, \alpha_s)dW_s$$

Stochastic control problem

$$V(t,x) = \inf_\alpha \mathbb{E}\Big[g(X_T^{t,x})\beta_{t,T} + \int_t^T \beta_{t,s} f(s, X_s^{t,x}, \alpha_s)ds \Big],$$

with $\beta_{t,s} := e^{-\int_t^s k(r, X_r^{t,x}, \alpha_r)dr}$

Under standard assumptions, V viscosity solution of the HJB equation

Comparison of viscosity sub- and supersolutions

Assume $U - V \leq 0$ on $\partial\mathcal{O}$ and $M := \max(U - V) > 0$. Then

$$\max(U - V) = (U - V)(x_0) \quad \text{for some interior point} \quad x_0 \in \mathcal{O}$$

- **In the smooth case**
  - $D(U - V)(x_0) = 0$ yields a contradiction for 1st order equations
  - 2nd order : use the second order condition $D^2(U - V)(x_0) \leq 0$ together with the ellipticity of the equation...
- **If U, V are only continuous** use doubling variables

$$M_n := \max_{\mathcal{O}\times\mathcal{O}} U(x) - V(y) - n|x - y|^2, \quad \text{attained at } (x_n, y_n) \in \mathcal{O} \times \mathcal{O}$$

$$\text{so that } n|x_n - y_n|^2 \to 0$$

$$\Longrightarrow \phi(x) := V(y_n) + n|x - y_n|^2 \text{ test function for } U \text{ at } x_n$$
$$\Longrightarrow \psi(y) := U(y_n) - n|x_n - y|^2 \text{ test functions for } V \text{ at } y_n$$

induces required contradiction for 1st order equations...

- Second order equations : needs to be complemented with measure theoretic arguments from the Crandall-Ishii's lemma...

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HJB equation on Hilbert spaces

- Extension by Lions, Swiech, Gozzi, ...

- Crandall-Lions and Li-Yong use

Test functions of the form $\varphi + \phi$ with **nonsmooth** ϕ

But, this is not the extension that we are exploring here...

We would like to analyze stochastic processes arising from optimal control theory by HJB-type of PDEs

HJB equation on the continuous paths space $C^0(\mathbb{R}_+, \mathbb{R}^d)$

Consider the non-Markov version of the previous control problem

$$X^{t,\omega}_{\wedge t} = \omega_{\wedge t} \text{ and } dX^{t,\omega}_s = b_s(X^{t,\omega}_{\wedge s}, \alpha_s)ds + \sigma_s(X^{t,\omega}_{\wedge s}, \alpha_s)dW_s$$

Stochastic control problem

$$V_t(\omega) = \sup_\alpha \mathbb{E}\Big[g(X^{t,\omega}_T)\beta_{t,T} + \int_t^T \beta_{t,s} f(s, X^{t,\omega}_s, \alpha_s)ds\Big],$$

with $\beta_{t,s} := e^{-\int_t^s k(r, X^{t,\omega}_r, \alpha_r)dr}$

- Ekren-NT-Zhang : Test functions $\mathbb{E}-$tangent
- Zhou : back to standard def (pointwise tangency) using Ekeland-Borwein-Preis variational Lemma

Under standard assumptions, V is the unique viscosity solution of the path-dependent HJB equation

$$\partial_t V + \inf_{a\in A}\Big\{b(.,a)\cdot\partial_\omega V + \frac{1}{2}\sigma\sigma^\intercal(.,a):\partial^2_{\omega\omega} V - k(.,a)V + f(.,a)\Big\} = 0$$

Dupire's derivatives on continuous paths space

For an adapted process $\{F_t, t \geq 0\}$, define the following derivatives (when limits exist) :

- Time derivative $\partial_t F(t,\omega) := \lim_{h\searrow 0} \frac{F_{t+h}(\omega_{\wedge t}) - F_t(\omega)}{h}$
- vertical derivative $\partial_\omega F(t,\omega) := \lim_{\varepsilon\to 0} \frac{F_t(\omega+\varepsilon \mathbf{1}_{[t,\infty)}) - F_t(\omega)}{\varepsilon}$

X : canonical proc. on the path space, i.e. $X_t(\omega) = \omega(t)$, $\omega \in C^0(\mathbb{R}_+, \mathbb{R}^d)$

Theorem (Dupire, Cont-Fournier)

Assume $\partial_t F$ and $\partial^2_{\omega\omega} F$ exist and continuous. Then the following Itô formula holds for any semimartingale measure $\mathbb{P}$ on the paths space :

$$dF_t = \partial_t F_t dt + \partial_\omega F_t \cdot dX_t + \frac{1}{2}\partial^2_{\omega\omega} F : d\langle X\rangle_t, \quad \mathbb{P} - \text{a.s.}$$

- Weakly $C^{1,2}$ smooth if Itô's formula holds under any semimart. meas.
- Corresponding Sobolev notion of solution through **backward SDEs**

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- Corresponding Sobolev notion of solution through **backward SDEs**

Finite population $X^1, \ldots, X^N$ driven by independent BMs W^i in $\mathbb{R}^d$

$$X_t^i = x^i \text{ and } dX_s^i = b(s, X_s^i, m_N(X_s), \alpha_s^i)ds + \sigma(s, X_s^i, \underbrace{m_N(X_s)}_{:=\frac{1}{N}\sum_{i=1}^N \delta_{X_s^i}}, \alpha_s^i)dW_s^i$$

Stochastic control problem

$$V_N(t,x) = \sup_{\alpha^1,\ldots,\alpha^N} \mathbb{E}\Big[\sum_{i=1}^N g(X_T^i, m_N(X_T))\beta_{t,T} + \int_t^T \beta_{t,s}^i f(s, X_s^i, \alpha_s^i)ds\Big],$$

with $\beta_{t,s}^i := e^{-\int_t^s k(r, X_r^i, \alpha_r)dr}$

Then HJB equation for this problem is

$$0 = \partial_t V_N + \sum_{i=1}^N \inf_{a\in A}\Big\{ b(x^i, m_N(x), a)\cdot D_{x^i} V_N + \frac{1}{2}\sigma\sigma^\intercal(x^i, m_N(x), a) : D^2_{x^i x^i} V_N$$

$$-k(x^i, m_N(x), a)V_N + f(x^i, m_N(x), a)\Big\}$$

Infinite population problem $N \to \infty$:

$$V_N(t,x) \longrightarrow V(t,m) := \sup_\alpha \mathbb{E}\Big[g(X_T,\mathcal{L}_{X_T})\beta_{t,s} + \int_t^T \beta_{t,s} f(s,X_s,\alpha_s)ds\Big]$$

where $\beta_{t,s} := e^{-\int_t^s k(r,X_r,\alpha_r)dr}$,

and X is defined by the controlled McKean-Vlasov SDE

$$\mathcal{L}_{X_t} = m \text{ and } dX_s = b(s,X_s,\mathcal{L}_{X_s},\alpha_s)ds + \sigma(s,X_s,\mathcal{L}_{X_s},\alpha_s)dW_s$$

V is a solution of the HJB equation on the Wasserstein space

$$0 = \partial_t V + \int \inf_{a\in A}\Big\{b(x,m,a)\cdot \partial_L V + \frac{1}{2}\sigma\sigma^\intercal(x,m,a) : \partial_x\partial_L V - k(x,m,a)V + f(x,m,a)\Big\} m(dx)$$

Mean field control with common noise

Suppose particles dynamics affected by a **common BM** W^0 **:**

$$X_t^i = x^i \text{ and } dX_s^i = b(s, X_s^i, m_N(X_s), \alpha_s^i)ds + \sigma(\cdots)dW_s^i + \sigma^0(\cdots)dW_s^0$$

Then, the corresponding mean field limit is :

$$\mathcal{L}(X_t) = m \text{ and } dX_s = b(s, X_s, \mathcal{L}_{X_s|W_0}, \alpha_s)ds + \sigma(\cdots)dW_s + \sigma^0(\cdots)dW_s^0$$

$$V(t,m) := \sup_{\alpha} \mathbb{E}\Big[g(X_T, \mathcal{L}_{X_s|W_0})\beta_{t,s} + \int_t^T \beta_{t,s} f(s, X_s, \alpha_s)ds\Big]$$

where $\beta_{t,s} := e^{-\int_t^s k(r,X_r,\alpha_r)dr}$,

Then, V solution of a second order HJB eq. on the Wasserstein space ...

Viscosity solution of PDEs on the Wasserstein space

- 1st order equation on the Wasserstein space :
  - involving $\partial_L u$ only : Bertucci, Cardaliaguet-Quincampoix, Conforti, Kraaij-Tonon, Feng-Katsoulakis, Gangbo, Nguyen-Tudorascu, Gangbo-Tudorascu, Jimenez, Marigonda-Quincampoix
  - involving $\partial_L u$ and $\partial_x \partial_L u$: Wu-Zhang, Cosso-Gozzi-Kharroubi-Pham-Rosestolato, Talbi-NT-Zhang, Burzoni-Ignazio-Reppen-Soner, Soner-Yan
- 2nd order equation on the Wasserstein space :
  - Bayraktar-Ekren-Zhang extend the Crandal Ishii's lemma to the present context
  - Gangbo-Mayorga-Swiech, Mayorga-Swiech, Daudin-Seeger : lifting on the Hilbert space of random variables

Our main objective is

Underlying space is the set of random processes : appears naturally and allows for more test functions, thus potentially helpful for uniqueness

See example 1 for intuition

Test functions will have a smooth component and a singular one

$$\varphi + \phi$$

Comparison will be obtained by (several) **doubling variables argument only**, thus avoiding the Crandall-Ishii lemma

See Example 2 below for intuition

Let W, W^0 indep. BM ob $(\Omega, \mathcal{F}, \mathbb{P})$ and consider the MF control pb.

$$\mathfrak{V}(0,\mu) := \inf_{\alpha} \mathbb{E}\big[\mathfrak{g}\big(\mathcal{L}_{X_T|\mathcal{F}_T^{W^0}}\big)\big], \text{ where } \mathfrak{g} : \mathcal{P}_2(\mathbb{R}^d) \longrightarrow \mathbb{R}$$

and

$$\mathcal{L}_{X_0|\mathcal{F}_T^{W^0}} = \mu$$

$$dX_t = \mathfrak{b}\big(X_t, \mathcal{L}_{X_t|\mathcal{F}_T^{W^0}}, \alpha_t\big)dt + \sigma^0 dW^0, \quad t \ge 0$$

Define $b : \Omega \times \mathbb{R}^d \times \mathbb{L}^2(\mathbb{R}^d) \times A \longrightarrow \mathbb{R}^d$ and $g : \Omega \times \mathbb{L}^2(\mathbb{R}^d) \longrightarrow \mathbb{R}$:

$$b(\omega, x, \underline{\xi}, a) := \mathfrak{b}\big(x, \mathcal{L}_{\xi|\mathcal{F}_t^{W^0}}(\omega), a\big) \text{ and } g(\omega, \underline{\xi}) := \mathfrak{g}\big(\mathcal{L}_{\xi|\mathcal{F}_t^{W^0}}(\omega)\big)$$

where we denote $\underline{\xi}$ to emphasize dependence on the r.v. ξ. Then

$$\mathfrak{V}(0,\mu) = V(0, X_0) := \inf_{\alpha} \mathbb{E}\big[g(\underline{X}_T)\big]$$

$$\text{where } dX_t = b\big(X_t, \underline{X}_t, \alpha_t\big)dt + \sigma^0 dW^0, \quad t \ge 0$$

$\Longrightarrow$ Control problem on the space of r.v.

Example 2 : constant diffusion setting

Consider the mean field control problem

$$\mathfrak{V}(0,\mu) \quad = \quad \inf_{\alpha} \mathfrak{g}\big(\mathcal{L}_{X_T}\big)$$

where $\mathcal{L}_{X_0} = \mu$ and $dX_t = b\big(X_t, \alpha_t\big)dt + dW, \quad t \geq 0$

Introducing the change of variable$x_t := X_t - W_t$, we see that

$$dx_t = f_t(x_t, \alpha_t)dt \quad \text{with} \quad f_t(\omega, x, a) := b(x + W_t(\omega), a)$$

Then $\mathfrak{V}_t(\underline{\xi}) = V_t(\underline{\xi} - \underline{W}_t)$, where

$$V_t(\underline{\xi}) := \inf_{\alpha} g(\underline{x}_T), \qquad g(\underline{x}_T) := \mathfrak{g}\big(\mathcal{L}_{x_T + W_T}\big)$$

where, here again $g : \mathbb{L}^2(\mathbb{R}^d) \longrightarrow \mathbb{R}$

$\Longrightarrow$ 1st order control problem... on the space of random variables

Doubling variables for the reduced 1st order problem

In the setting of the first order control problem, it is natural to adapt the doubling variables argument with test functions

$$\varphi(t,\underline{\xi}) \quad := \quad n\Big(|t-s_n|^2+\mathbb{E}[|\xi-\zeta_n|^2]\Big)+\cdots$$

Recalling our change of variable, this induces the following test function for the initial problem :

$$\phi(t,\underline{\xi}) \quad := \quad n\Big(|t-s_n|^2+\mathbb{E}[|\xi-W_t-\zeta_n|^2]\Big)+\cdots$$

$\Longrightarrow$ introduces a dependence on joint law of (ξ,ζ_n,W)

$\Longrightarrow$ φ is smooth, BUT ϕ is not $C^{1,2}$... (see later in which sense) ; However, it is absolutely continuous...

The above ϕ is our typical singular component of test function

Extending Example 2 to nonconstant diffusion coefficient

What about the mean field control problem

$$\mathfrak{V}(0,\mu) \quad = \quad \inf_{\alpha} \mathfrak{g}\big(\mathcal{L}_{X_T}\big)$$

where $\mathcal{L}_{X_0} = \mu$ and $dX_t = b\big(X_t, \alpha_t\big)dt + \sigma\big(X_t, \alpha_t\big)dW, \quad t \geq 0$

- It is not possible anymore to perform the direct change of variable
- but it turn out that we can act on test functions through the singular component
- The price to pay is to introduce a further dependence on the process

$$\underline{\mathrm{X}} = \{X_t, t \in [0, T]\}$$

and not only on the random variables $\underline{X}_t$, $t \in [0, T]$

The control problem on the space of random processes

We shall consider the control problem on the space of processes

$$V_t(\underline{\xi}) := \inf_{\alpha \in \mathcal{A}_{[t,T]}} g(\underline{X}^{t,\xi,\alpha}) + \int_t^T f_s(\underline{X}^{t,\xi,\alpha}, \underline{\alpha_s}) ds$$

where the controlled state is defined by

$$X^{t,\xi,\alpha}_{\wedge t} = \xi, \quad \text{and for } s \geq t:$$

$$dX^{t,\xi,\alpha}_s = b_s\big(X^{t,\xi,\alpha}, \alpha_s, \underline{X}^{t,\xi,\alpha}, \underline{\alpha_s}\big) ds + \sigma_s\big(X^{t,\xi,\alpha}, \alpha_s, \underline{X}^{t,\xi,\alpha}, \underline{\alpha_s}\big) dW_s$$

All above functions are maps :

$$[0,T] \times \Omega \times \mathbb{R}^d \times \mathbb{S}^2 \times \mathbb{H}^2 \longrightarrow \mathbb{R}^d, \mathbb{R}d \times d, \text{ or } \mathbb{R}$$

$\mathbb{S}^p$: continuous $\mathbb{F}-$p.m. proc. with $\|X\|_{\mathbb{S}^p} = \||X|_\infty\|_{\mathbb{L}^p(\mathbb{R})} < \infty$
$\mathbb{H}^2$: $\mathbb{F}-$p.m. proc. with $\|\alpha\|_{\mathbb{H}^2} = \|\alpha\|_{\mathbb{L}^2([0,T]\times A)} < \infty$

Proposition

Under standard Lipschitz conditions, V is uniformly Lipschitz in ξ, and locally $\frac{1}{2}-$Hölder continuous in t :

$$\big|V_t(\underline{\xi}) - V_{t'}(\underline{\xi}')\big| \leq C\|\xi_{\wedge t} - \xi'_{\wedge t}\|_{\mathbb{S}^2} + C\big(1 + \|\xi_{\wedge t}\|_{\mathbb{S}^2}\big)|t - t'|^{\frac{1}{2}}$$

Smooth maps on the space of processes $\mathbb{S}^2$

Definition

A map $\varphi : [t, T] \times \mathbb{S}^p \to \mathbb{R}$ is $C^{1,2}$ if $\varphi \in C^0$ and there exist C^0 maps

$$\partial_t \varphi : [t,T] \times \mathbb{S}^p \to \mathbb{R} \quad \text{and} \quad \partial_X \varphi : [t,T] \times \mathbb{S}^p \longrightarrow \mathbb{L}^{\frac{p}{p-1}}(\mathbb{R}^d)$$
$$\partial_{xX} \varphi : [t,T] \times \mathbb{S}^p \longrightarrow \mathbb{L}^{\frac{p}{p-1}}(\mathbb{R}^{d\times d})$$

such that $\partial_X \varphi_t(\underline{\xi})$, $\partial_X \varphi_t(\underline{\xi})$ $\mathcal{F}_t-$meas. and for all Itô process X

$$d\varphi_t(\underline{X}) = \partial_t \varphi_t(\underline{X}) dt + \mathbb{E}\Big[\partial_X \varphi_t(\underline{X}) \cdot dX_t + \frac{1}{2} \partial_{xX} \varphi_t(\underline{X}) : d\langle X \rangle_t\Big]$$

- Consistent with L-derivative in the law-invariant setting
- Holds for smooth functions in the Fréchet-Dupire sense...

Define

- time derivative : $\lim_{h\searrow 0} \frac{1}{h}\left[\varphi_{t+h}(\underline{\xi}_{\wedge t}) - \varphi_t(\underline{\xi})\right]$, if exists
- Space derivative : $\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\left[\varphi_t(\underline{\xi}+\varepsilon\underline{\zeta}) - \varphi_t(\underline{\xi})\right] =: \langle \partial_X\varphi_t(\underline{\xi}), \zeta\rangle_{\mathbb{L}^2}$, if exists

Assume that such derivatives exist and continuous, then for an Itô process X, we estimate :

$$\varphi_{t_{i+1}}(\underline{X}_{t_{i+1}}) - \varphi_{t_i}(\underline{X}_{t_i}) = \underbrace{\varphi_{t_{i+1}}(\underline{X}_{t_{i+1}}) - \varphi_{t_{i+1}}(\underline{X}_{t_i})}_{=\mathbb{E}[\partial_X\varphi_{t_{i+1}}(\underline{\xi}_{i+1})(X_{t_{i+1}} - X_{t_i})]} + \underbrace{\varphi_{t_{i+1}}(\underline{X}_{t_i}) - \varphi_{t_i}(\underline{X}_{t_i})}_{=\partial_t\varphi_{\tau_{i+1}}(\underline{X}_{t_i})(t_{i+1}-t_i)}$$

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Actually need two space derivatives

Then $\sum_i\cdots$ and send time step to 0...

HJB equation on the space of processes $\mathbb{S}^2$

Combining the dynamic programming principle with the last Itô formula, we obtain the following dynamic programming equation on $\mathbb{Q}_0^2 := [0, T] \times \mathbb{S}^2$:

$$\partial_t U_t(\underline{\xi}) + \inf_{\alpha} H_t\big(\underline{\xi}, \partial_X U_t(\underline{\xi}), \partial_{xX} U_t(\underline{\xi}), \underline{\alpha}\big) = 0$$

$$\text{where } H_t\big(\underline{\xi}, \underline{Z}, \underline{\Gamma}, \underline{\alpha}\big) := \mathbb{E}\Big[\Big(b_t \cdot Z + \frac{1}{2}\sigma\sigma_t^\top : \Gamma\Big)(\xi, \alpha, \underline{\xi}, \underline{\alpha}) + f_t(\underline{\xi}, \underline{\alpha})\Big]$$

Theorem (Characterization of value function)

Under our assumptions, the value function V is the unique viscosity solution of the HJB equation with terminal condition $V_T = g$ in the class of functions satisfying

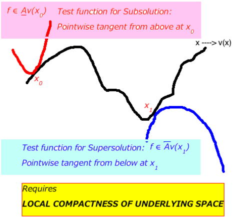
$$\left|V_t(\underline{\xi}) - V_{t'}(\underline{\xi}')\right| \leq C\|\xi_{\wedge t} - \xi'_{\wedge t}\|_{\mathbb{S}^2} + C\big(1 + \|\xi_{\wedge t}\|_{\mathbb{S}^2}\big)|t - t'|^{\frac{1}{2}}$$

Theorem (Comparison result)

Let $U^0, U^1 \in C^0(\mathbb{Q}_0^2)$ be viscosity subsolution and supersolution, respectively, of the HJB equation satisfying

$$\left|U^i_t(\underline{\xi}) - U^i_{t'}(\underline{\xi}')\right| \leq C\|\xi_{\wedge t} - \xi'_{\wedge t}\|_{\mathbb{S}^2} + C\big(1 + \|\xi_{\wedge t}\|_{\mathbb{S}^2}\big)|t - t'|^{\frac{1}{2}}$$

Then, under our assumptions, $U^0_T \leq U^1_T$ on $\mathbb{Q}_0^2$ implies $U^0 \leq U^1$ on $\mathbb{S}^2$



For $U \in C^0(\mathbb{Q}_0^2)$ and (t, ξ)

$$\mathfrak{F}^+ U_t(\underline{\xi}) := \Big\{ (\varphi, \phi) \in C^{1,2}(\mathbb{Q}_t^6) \times C^+(\mathbb{Q}_t^6) : \big[U - (\varphi + \phi)\big]_t(\underline{\xi}) = \sup_{\mathbb{Q}_t^6} \big[U - (\varphi + \phi)\big] \Big\}$$

$$\mathfrak{F}^- U_t(\underline{\xi}) := \Big\{ (\varphi, \phi) \in C^{1,2}(\mathbb{Q}_t^6) \times C^-(\mathbb{Q}_t^6) : \big[U - (\varphi + \phi)\big]_t(\underline{\xi}) = \inf_{\mathbb{Q}_t^6} \big[U - (\varphi + \phi)\big] \Big\}$$

Given $(s, \zeta) \in \mathbb{Q}_0^p$ and continuous $(\beta, \gamma) : A \times \mathcal{A} \to \mathbb{R}^d, \mathbb{R}^{d\times d}$, denote :

$$\mathcal{I}_t^\alpha(\xi) := \mathcal{I}_t^{\beta,\gamma,s,\zeta,\alpha}(\xi) := \xi_t - \zeta_s - \int_s^t \beta_r^\alpha dr - \int_s^t \gamma_r^\alpha dW_r$$

where $\beta_r^\alpha := \beta(\alpha_r, \underline{\alpha}_r)$

Definition

For $s \in [0, T]$, we denote $C^+(\mathbb{Q}_s^p)$ the set of maps of the form :

$$\phi_t(\underline{\xi}) := \inf_{\alpha\in\mathcal{A}_{[s,T]}} \Big\{ k\mathbb{E}\Big[\big|\mathcal{I}_t^\alpha(\xi)\big|^p + \big|\mathcal{I}_{t'}^\alpha(\xi')\big|^p\Big] + \int_{t'}^t \psi_r^\alpha dr \Big\} \text{ for all } (t,\xi) \in \mathbb{Q}_s^p$$

for some $k \geq 0$, $\zeta \in \mathbb{S}^p$, $(t', \xi') \in \mathbb{Q}_s^p$, β, γ as above, $\psi \in C^0(\mathcal{A}, \mathbb{R})$.

Moreover, let $C^-(\mathbb{Q}_s^p) := -C^+(\mathbb{Q}_s^p)$

Fact : Any $\phi \in C^+(\mathbb{Q}_s^p)$ is a.c. wrt Lebesgue

Frozen state process defined for all (t,ξ) and α

$$\bar{X}_s^{t,\xi,\alpha} := \xi_t + \mathbb{1}_{\{s\geq t\}} \int_t^s b_r^{t,\xi,\alpha} dr + \int_t^s \sigma_r^{t,\xi,\alpha} dW_r, \;\; s \geq 0,$$

where $\psi_s^{t,\xi,\alpha} := \psi(t, \xi_{\wedge t}, \alpha_s, \underline{\xi}_{\wedge t}, \underline{\alpha}_s)$ for $\psi = b, \sigma$

Definition

(i) $U \in C^0(\mathbb{Q}_0^2)$ is a viscosity subsolution of HJB equation if

$$\partial_t \varphi_t(\underline{\xi}) + \liminf_{\delta\to 0} \inf_\alpha \frac{1}{\delta} \int_t^{t+\delta} \Big[H_s(\underline{\xi}_{\wedge t}, \partial_X \varphi_t(\underline{\xi}), \partial_{xX} \varphi_t(\underline{\xi}), \underline{\alpha}_s) + \dot{\phi}_s(\underline{\bar{X}}^{t,\xi,\alpha}) \Big] ds \geq 0$$

for all $(t,\xi) \in \mathbb{Q}_0^6$ and $(\varphi, \phi) \in \mathfrak{F}^+ U_t(\underline{\xi})$

(ii) $U \in C^0(\mathbb{Q}_0^2)$ is a viscosity supersolution of HJB equation if

$$\partial_t \varphi_t(\xi) + \limsup_{\delta\to 0} \inf_\alpha \frac{1}{\delta} \int_t^{t+\delta} \Big[H_s(\underline{\xi}_{\wedge t}, \partial_X \varphi_t(\underline{\xi}), \partial_{xX} \varphi_t(\underline{\xi}), \underline{\alpha}_s) + \dot{\phi}_s(\underline{\bar{X}}^{t,\xi,\alpha}) \Big] ds \leq 0$$

for all $(t,\xi) \in \mathbb{Q}_0^6$ and $(\varphi, \phi) \in \mathfrak{F}^- U_t(\underline{\xi})$

(iii) $U \in C^0(\mathbb{Q}_0^2)$ is a viscosity solution of HJB if ...