Multitask ECODEP Conference
Paris, IHP, 2024 02 12-14

Fractionally integrated spatial models and statistical applications

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Contents:

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1. Fractional integration in dimension $\nu = 1$. LRD & fractional processes
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$$(1-z)^d = \sum_{j=0}^{\infty} \psi_j(d) z^j, \qquad \psi_j(d) := \frac{\Gamma(j-d)}{\Gamma(j+1)\Gamma(-d)}, \quad z \in \mathbb{C}, \ |z| < 1.$$

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Namely,

$$(I-T)^d g(t) := \sum_{j=0}^{\infty} \psi_j(d) T^j g(t) = \sum_{j=0}^{\infty} \psi_j(d) g(t-j), \qquad t \in \mathbb{Z}$$

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- Commutative group: $(I-T)^{d_1}(I-T)^{d_2} = (I-T)^{d_1+d_2}$ $(|d_1|, |d_2|, |d_1+d_2| < 1)$, $(I-T)^d (I-T)^{-d} = I$

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- Autoregressive fractionally integrated moving-average (ARFIMA $(0, d, 0)$) process $X = \{X(t); t \in \mathbb{Z}\}$ defined as the solution of the stochastic difference equation

$$(I - T)^d X(t) = \sum_{j=0}^{\infty} \psi_j(d) X(t-j) \; = \; \varepsilon(t), \qquad t \in \mathbb{Z} \tag{1}$$

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where $\{\varepsilon(t); t \in \mathbb{Z}\}$ is an i.i.d. (white noise), with zero mean and finite variance. For $d \in (-1/2, 1/2), d \neq 0$ the unique stationary solution of (1) or ARFIMA $(0, d, 0)$ process writes as MA process:

$$X(t) \; = \; (I - T)^{-d} \varepsilon(t) \; = \; \sum_{j=0}^{\infty} \psi_j(-d) \varepsilon(t-j)$$

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- $D^{\alpha}I^{\alpha} = I$
- Fractionally integrated white noise $\dot{B}(t) := \mathrm{d}B(t)/\mathrm{d}t$ is FBM with $H = \alpha + 1/2 \in (0, 1)$

$$X(t) := \begin{cases} \int_0^t (I^{\alpha}\dot{B})(s)\mathrm{d}s, & 0 < \alpha < 1/2, \\ \int_0^t (D^{\alpha}\dot{B})(s)\mathrm{d}s, & -1/2 < \alpha < 0 \end{cases}$$

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- Fractional powers $(I-T)^d, -1 < d < 1$ can be defined similarly to $\nu = 1$ through binomial expansion $(1-z)^d = \sum_{j=0}^\infty \psi_j(d)z^j, |z|<1$:

$$(I-T)^d g(\boldsymbol{t}) := \sum_{j=0}^\infty \psi_j(d)T^j g(\boldsymbol{t}) = \sum_{\boldsymbol{s}\in\mathbb{Z}^\nu} \tau(\boldsymbol{s};d)g(\boldsymbol{t}-\boldsymbol{s}), \quad \text{where}$$

$$\tau(\boldsymbol{s};d) := \sum_{j=0}^\infty \psi_j(d)p_j(\boldsymbol{s}), \quad \boldsymbol{s}\in\mathbb{Z}^\nu \tag{2}$$

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- Fractional powers $(I-T)^d, -1 < d < 1$ can be defined similarly to $\nu = 1$ through binomial expansion $(1-z)^d = \sum_{j=0}^\infty \psi_j(d)z^j, |z| < 1$:

$$(I-T)^d g(\mathbf{t}) := \sum_{j=0}^\infty \psi_j(d) T^j g(\mathbf{t}) = \sum_{\mathbf{s}\in\mathbb{Z}^\nu} \tau(\mathbf{s}; d) g(\mathbf{t}-\mathbf{s}), \quad \text{where}$$

$$\tau(\mathbf{s}; d) := \sum_{j=0}^\infty \psi_j(d) p_j(\mathbf{s}), \quad \mathbf{s}\in\mathbb{Z}^\nu \tag{2}$$

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Example 3. $\nu \geq 2$, fractional heat operator $(I - T)^d = (\Delta_{1,2})^d$:

$$\begin{aligned} \Delta_{1,2} g(\boldsymbol{t}) \quad &:= \quad (1 - \theta)(g(\boldsymbol{t}) - g(\boldsymbol{t} - \boldsymbol{e}_1)) \\ &- \quad \frac{\theta}{2(\nu - 1)} \sum_{j=2}^{\nu} (g(\boldsymbol{t} - \boldsymbol{e}_1 + \boldsymbol{e}_j) + g(\boldsymbol{t} - \boldsymbol{e}_1 - \boldsymbol{e}_j) - 2g(\boldsymbol{t})). \end{aligned}$$

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Example 4. Unilateral fractional operators $(I - T_1)^{d_1} \cdots (I - T_\nu)^{d_\nu}$, $T_j g(\boldsymbol{t}) := g(\boldsymbol{t} - \boldsymbol{e}_j)$, $j = 1, \cdots, \nu$

Main object: Fractionally integrated random field (RF) X defined as solution of

$$(I - T)^d X(\boldsymbol{t}) = \varepsilon(\boldsymbol{t}), \quad \boldsymbol{t} \in \mathbb{Z}^\nu, \tag{3}$$

with i.i.d. white noise $\{\varepsilon(\boldsymbol{t})\}$ written as MA

$$X(\boldsymbol{t}) = (I - T)^{-d}\varepsilon(\boldsymbol{t}) = \sum_{\boldsymbol{s}\in\mathbb{Z}^\nu} \tau(\boldsymbol{s}; -d)\varepsilon(\boldsymbol{t} - \boldsymbol{s}), \quad \boldsymbol{t} \in \mathbb{Z}^\nu. \tag{4}$$

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2. Fractional integration on $\mathbb{Z}^\nu$. Examples

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$$\int_{\Pi^\nu} |1 - \widehat{p}(\boldsymbol{x})|^{-2|d|} \mathrm{d}\boldsymbol{x} < \infty \tag{5}$$

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LRD asymptotics of fractional coefficients $\tau(\boldsymbol{s}; d), |\boldsymbol{s}| \to \infty$. Assume 'typical' conditions for local CLT:

$$\mathrm{E}\mathrm{e}^{c|S_1|} < \infty \quad (\exists\, c > 0) \quad \text{and } \{S_j\} \text{ is zero mean, aperiodic, irreducible.} \tag{6}$$

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(6) imply that RW has invertible covariance matrix

$$\Gamma := \mathrm{E} S_1 S_1' = \Lambda \Lambda'$$

and $\Lambda^{-1} S_1$ has unit covariance matrix.

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Let (6) *hold. Then* $\tau(\boldsymbol{s}; d)$ *are well-defined for any* $-(1 \wedge \frac{\nu}{2}) < d < 1, d \neq 0$ *and satisfy*

$$\tau(\boldsymbol{s}; d) = (B_1(d) + o(1))(\boldsymbol{s} \cdot \Gamma^{-1}\boldsymbol{s})^{-(\nu/2)-d}, \qquad |\boldsymbol{s}| \to \infty,$$

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- Γ unit matrix: isotropic decay $\tau(\mathbf{s}; d) \sim const.|\mathbf{s}|^{-\nu-2d}$

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where $B_1(d) := \frac{2^d \Gamma(d+\frac{\nu}{2})}{\pi^{\nu/2}\Gamma(-d)\sqrt{\det \Gamma}}$.

- Γ unit matrix: isotropic decay $\tau(\mathbf{s}; d) \sim const.|\mathbf{s}|^{-\nu-2d}$
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Theorem (2)

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- Γ unit matrix: isotropic decay $\tau(\boldsymbol{s};d)\sim const.|\boldsymbol{s}|^{-\nu-2d}$
- Thm does not apply to ARFIMA$(0,d,0)$ and fractional heat operator because of nonzero mean RW
- fractional heat operator $\tau(\boldsymbol{s};d)$ satisfy anisotropic asymptotics

$$\tau(\boldsymbol{s};d)=\frac{s_1^{-d-\frac{1+\nu}{2}}}{\Gamma(d)(2\pi\theta)^{(\nu-1)/2}\sqrt{\det\tilde{\Gamma}}}\exp\Big\{-\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\Gamma}^{-1}\tilde{\boldsymbol{s}}}{2\theta s_1}\Big\}\big(1+o(1)\big), \quad \boldsymbol{s}=(s_1,\tilde{\boldsymbol{s}})\in\mathbb{Z}^\nu$$

Pilipauskaitė & S. (2017), S. (2020)

3. Fractionally integrated RFs on $\mathbb{R}^\nu$

- $I - T$ a local (differential) operator

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Example 5. (Nonstationary) *Fractional Brownian/Lévy RF* with parameter $H \in (0, 1), H \neq \nu/2$ is usually defined as stochastic integral

$$\mathcal{B}_H(\boldsymbol{t}) := \int_{\mathbb{R}^\nu} \left(|\boldsymbol{t} + \boldsymbol{u}|^{H - \frac{\nu}{2}} - |\boldsymbol{u}|^{H - \frac{\nu}{2}} \right) M(\mathrm{d}\boldsymbol{u}), \quad \boldsymbol{t} \in \mathbb{R}^\nu$$

w.r.t. Gaussian/Lévy random measure $M(\mathrm{d}\boldsymbol{u})$ with zero mean and finite variance

- $\mathrm{E}\mathcal{B}_H(\boldsymbol{t})\mathcal{B}_H(\boldsymbol{s}) = \mathrm{const.}\big(|\boldsymbol{t}|^{2H} + |\boldsymbol{s}|^{2H} - |\boldsymbol{t}-\boldsymbol{s}|^{2H}\big)$

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- Solves $(-\Delta)^{\frac{H}{2}+\frac{\nu}{4}}\mathcal{B}_H(\boldsymbol{t}) = \mathrm{const.}\dot{M}(\boldsymbol{t})$ with fractional Laplacian, $\dot{M}(\boldsymbol{t}) = M(\mathrm{d}\boldsymbol{t})/\mathrm{d}\boldsymbol{t}$

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Example 6. (Stationary) *Matérn RF* with parameters $c, H > 0$ defined as

$$\mathcal{M}_{c,H}(\boldsymbol{t}) := \int_{\mathbb{R}^\nu} m_{c,H}(\boldsymbol{t}-\boldsymbol{u})M(\mathrm{d}\boldsymbol{u}), \qquad \boldsymbol{t} \in \mathbb{R}^\nu,$$

where

$$m_{c,H}(\boldsymbol{t}) := \mathrm{const.}|c\boldsymbol{t}|^{\frac{H}{2}-\frac{\nu}{4}} K_{\frac{H}{2}-\frac{\nu}{4}}(c|\boldsymbol{t}|), \quad \boldsymbol{t} \in \mathbb{R}^\nu,$$

K_τ = modified Bessel function, M the same as in Example 5

3. Fractionally integrated RFs on $\mathbb{R}^\nu$

- (Matérn) covariance function:

$$\mathrm{E}\mathcal{M}_{c,H}(\mathbf{0})\mathcal{M}_{c,H}(\boldsymbol{t}) = \mathrm{const.}(c|\boldsymbol{t}|)^H K_H(c|\boldsymbol{t}|), \quad \boldsymbol{t} \in \mathbb{R}^\nu$$

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Example 7. (Stationary) *fractional heat operator RF* with parameters $c > 0, d > \frac{\nu+1}{4}$:

$$\mathcal{H}_{c,d}(\boldsymbol{t}) := \int_{\mathbb{R}^\nu} h_{c,d}(\boldsymbol{t} - \boldsymbol{u})M(\mathrm{d}\boldsymbol{u}), \qquad \boldsymbol{t} \in \mathbb{R}^\nu,$$

is defined in Kelbert, Leonenko & Ruiz-Medina (2005) as the RF with spectral density

$$f(\boldsymbol{z}) = |\widehat{h}_{c,d}(\boldsymbol{z})|^2 = \frac{1}{(z_1^2 + (c + |\tilde{\boldsymbol{z}}|^2)^2)^d}, \quad \boldsymbol{z} = (z_1, \tilde{\boldsymbol{z}}) \in \mathbb{R}^\nu,$$

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The MA kernel $h_{c,d}(\boldsymbol{t})$ was recently found in Pilipauskaitė & S. (2022, Bernoulli):

$$h_{c,d}(\boldsymbol{t}) = \mathrm{const.}t_1^{d-\frac{1+\nu}{2}} \exp\Big\{-ct_1 - \frac{|\tilde{\boldsymbol{t}}|^2}{4t_1}\Big\}\mathbf{1}(t_1 > 0), \quad \boldsymbol{t} = (t_1, \tilde{\boldsymbol{t}}) \in \mathbb{R}^\nu \qquad (7)$$

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3. Fractionally integrated RFs on $\mathbb{R}^\nu$

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Discretely fractionally integrated RFs in $\mathbb{R}^\nu$

Since fractional RFs in Examples 5-7 are SRD or nonstationary, we can define stationary LRD RFs by applying to them 'discrete' fractional integration/differentiation operators as discussed in sec.2

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Let

$$T_B g(\boldsymbol{t}) := \int_{\mathbb{R}^\nu} p_1(\boldsymbol{s} - \boldsymbol{t}) g(\boldsymbol{s}) \mathrm{d}\boldsymbol{s}, \quad \boldsymbol{t} \in \mathbb{R}^\nu \tag{8}$$

be the transition operator of a (discrete-time) standard Brownian random walk $\{B_j; j \in \mathbb{N}\}$ on $\mathbb{R}^\nu$ with Gaussian jth step transition probabilities

$$p_j(\boldsymbol{s} - \boldsymbol{t}) := (2\pi j)^{-\nu/2} \mathrm{e}^{-|\boldsymbol{s} - \boldsymbol{t}|^2/2j}, \quad \boldsymbol{t}, \boldsymbol{s} \in \mathbb{R}^\nu.$$

3. Fractionally integrated RFs on $\mathbb{R}^\nu$

T_B in (8) is well-defined for each $g \in L^p(\mathbb{R}^\nu), p \geq 1$ and $T_B^j g(\boldsymbol{t}) = \int_{\mathbb{R}^\nu} p_j(\boldsymbol{s} - \boldsymbol{t}) g(\boldsymbol{s}) \mathrm{d}\boldsymbol{s}$, $j = 0, 1, 2, \cdots$.

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Define

$$(I - T_B)^\kappa g(\boldsymbol{t}) \; := \; \int_{\mathbb{R}^\nu} \tau_B(\boldsymbol{s}; \kappa) g(\boldsymbol{s} + \boldsymbol{t}) \mathrm{d}\boldsymbol{s}, \quad \boldsymbol{t} \in \mathbb{R}^\nu, \tag{9}$$

with kernel

$$\tau_B(\boldsymbol{s}; \kappa) \; := \; \sum_{j=0}^\infty \psi_j(\kappa) p_j(\boldsymbol{s}), \quad \boldsymbol{s} \in \mathbb{R}^\nu \tag{10}$$

involving binomial coefficients $(1 - z)^\kappa = \sum_{j=0}^\infty z^j \psi_j(\kappa)$ as in sec.2.

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The ‘continuous’ kernel in (10) satisfies similar LRD/ND properties as the ‘discrete’ one in sec.2:

$$\tau_B(\boldsymbol{s}; \kappa) \sim \mathrm{const.} |\boldsymbol{s}|^{-\nu - 2\kappa}, \quad |\boldsymbol{s}| \to \infty, \quad -(1 \wedge \frac{\nu}{2}) < \kappa < 1, \kappa \neq 0$$

$$\int_{\mathbb{R}^\nu} \tau_B(\boldsymbol{s}; \kappa) \mathrm{d}\boldsymbol{s} = 0, \quad \kappa > 0,$$

$\tau_B(\boldsymbol{s}; \kappa)$ bdd & isotropic in $\boldsymbol{s} \in \mathbb{R}^\nu$

Fourier tr.: $\widehat{\tau}_B(\boldsymbol{z}; \kappa) = \sum_{j=0}^\infty \psi_j(\kappa) \mathrm{e}^{-j|\boldsymbol{z}|^2/2} = (1 - \mathrm{e}^{-|\boldsymbol{z}|^2/2})^\kappa, \quad \boldsymbol{z} \in \mathbb{R}^\nu.$

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The ‘continuous’ kernel in (10) satisfies similar LRD/ND properties as the ‘discrete’ one in sec.2:

$$\tau_B(\boldsymbol{s};\kappa) \sim \mathrm{const.}|\boldsymbol{s}|^{-\nu - 2\kappa}, \quad |\boldsymbol{s}| \to \infty, \quad -(1 \wedge \frac{\nu}{2}) < \kappa < 1, \kappa \neq 0$$

$$\int_{\mathbb{R}^\nu} \tau_B(\boldsymbol{s};\kappa) \mathrm{d}\boldsymbol{s} = 0, \quad \kappa > 0,$$

$\tau_B(\boldsymbol{s};\kappa)$ bdd & isotropic in $\boldsymbol{s} \in \mathbb{R}^\nu$

Fourier tr.: $\widehat{\tau}_B(\boldsymbol{z};\kappa) = \sum_{j=0}^\infty \psi_j(\kappa) \mathrm{e}^{-j|\boldsymbol{z}|^2/2} = (1 - \mathrm{e}^{-|\boldsymbol{z}|^2/2})^\kappa, \quad \boldsymbol{z} \in \mathbb{R}^\nu.$

3. Fractionally integrated RFs on $\mathbb{R}^\nu$

Fractional operator $(I - T_B)^\kappa$ cannot be applied to white noise $\dot{M}$ in $\mathbb{R}^\nu$ rather than to more regular RFs such as Brownian/Lévy RF or Matérn RF, yielding stationary RF with LRD:

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$$X(\boldsymbol{t}) \quad := \quad (I - T_B)^\kappa \mathcal{B}_H(\boldsymbol{t}) = \int_{\mathbb{R}^\nu} a(\boldsymbol{t} - \boldsymbol{u}) M(\mathrm{d}\boldsymbol{u}), \tag{11}$$

where $\kappa, H > 0$ and

$$a(\boldsymbol{t}) := \int_{\mathbb{R}^\nu} \tau_B(\boldsymbol{s}; \kappa)(|\boldsymbol{s} + \boldsymbol{t}|^{H - \frac{\nu}{2}} - |\boldsymbol{t}|^{H - \frac{\nu}{2}}) \mathrm{d}\boldsymbol{s}, \quad \boldsymbol{t} \in \mathbb{R}^\nu.$$

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- explicit spectral density

$$f(\boldsymbol{z}) = \frac{(1 - \mathrm{e}^{-|\boldsymbol{z}|^2/2})^{2\kappa}}{|\boldsymbol{z}|^{\nu + 2H}} \sim 1/|\boldsymbol{z}|^{\nu + 2H - 4\kappa} \to \infty \ (|\boldsymbol{z}| \to 0)$$

Example 9. Discretely fractionally integrated Matérn RF defined as

$$X(\boldsymbol{t}) \quad := \quad (I - T_B)^{-\kappa}\mathcal{M}_{c,H}(\boldsymbol{t}) = \int_{\mathbb{R}^\nu} a(\boldsymbol{t}-\boldsymbol{u})M(\mathrm{d}\boldsymbol{u}), \tag{12}$$

where $c, \kappa, H > 0$ and

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(or respective sums in the discrete argument case), where $X = \{X(\boldsymbol{t}); \boldsymbol{t} \in \mathbb{R}^\nu\}$ is a given stationary RF, *for each* ϕ from a linear class of (test) functions $\Phi = \{\phi : \mathbb{R}^\nu \to \mathbb{R}\}$

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  which contains indicator functions $\phi(\boldsymbol{t}) = \mathbb{I}(\boldsymbol{t} \in A)$ of arbitrary Borel sets of $A \subset \mathbb{R}^\nu, \mathrm{Leb}_\nu(A) < \infty$

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- Spatial statistics: accent on *irregular* (inflated) observation set $\lambda A \subset \mathbb{R}^\nu$ (rectangles not suffice)

  Lahiri & Robinson, Central limit theorems for long range dependent spatial linear processes (2016, Bernoulli)

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- Dependence properties of linear RF (16)/ (17) determined by MA kernel $a(\boldsymbol{t})$

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Assumption (A)$(d; \mathbb{Z}^\nu)$

(i) Let $0 < d < \nu/4$. Then

$$a(\boldsymbol{t}) \;=\; \frac{1}{|\boldsymbol{t}|^{\nu-2d}}\Big(\ell\big(\frac{\boldsymbol{t}}{|\boldsymbol{t}|}\big) + o(1)\Big), \quad |\boldsymbol{t}| \to \infty, \tag{18}$$

where $\ell(\boldsymbol{t}), |\boldsymbol{t}| = 1$ is a continuous 'angular' function

(ii) Let $-\nu/4 < d < 0$. Then (18) holds and, moreover, $\sum_{\boldsymbol{t}\in\mathbb{Z}^\nu} a(\boldsymbol{t}) = 0$.

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- Assumption (A)$(d; \mathbb{R}^\nu)$ (for 'cnt. arg. $a(\boldsymbol{t}), \boldsymbol{t} \in \mathbb{R}^\nu$) is analogous with sums in (ii), (iii) replaced by integrals $\int_{\mathbb{R}^\nu}$, and assuming boundedness of $a(\boldsymbol{t})$

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- Assumption (A)$(d; \mathbb{R}^\nu)$ (for 'cnt. arg. $a(\boldsymbol{t}), \boldsymbol{t} \in \mathbb{R}^\nu$) is analogous with sums in (ii), (iii) replaced by integrals $\int_{\mathbb{R}^\nu}$, and assuming boundedness of $a(\boldsymbol{t})$
- Define homogeneous limit function

$$a_\infty(\boldsymbol{t}) := |\boldsymbol{t}|^{2d-\nu}\ell\big(\tfrac{\boldsymbol{t}}{|\boldsymbol{t}|}\big), \quad \boldsymbol{t} \in \mathbb{R}^\nu_0 := \mathbb{R}^\nu \setminus \{\mathbf{0}\}$$

Limit Gaussian RFs written as stochastic integrals w.r.t. Gaussian WN $W(\mathrm{d}\boldsymbol{u})$:

$$W_d(\phi) := \begin{cases} \int_{\mathbb{R}^\nu} (a_\infty \star \phi)(\boldsymbol{u}) W(\mathrm{d}\boldsymbol{u}), & 0 < d < \nu/4, \phi \in \Phi \\ \int_{\mathbb{R}^\nu} (a_\infty \star \phi)_{\mathrm{reg}}(\boldsymbol{u}) W(\mathrm{d}\boldsymbol{u}), & -\nu/4 < d < 0, \phi \in \Phi_d^-, \\ \int_{\mathbb{R}^\nu} \phi(\boldsymbol{u}) W(\mathrm{d}\boldsymbol{u}), & d = 0, \phi \in \Phi, \end{cases} \tag{19}$$

- $\Phi_d^- := \Big\{ \phi \in \Phi,\ \phi(\cdot)\ \mathrm{a.e.cnt.}, \int_{\mathbb{R}^\nu} \big(\int_{\mathbb{R}^\nu} |\phi(\boldsymbol{t}+\boldsymbol{s}) - \phi(\boldsymbol{s})|^2 \mathrm{d}\boldsymbol{s} \big)^{1/2} |\boldsymbol{t}|^{2d-\nu} \mathrm{d}\boldsymbol{t} < \infty \Big\}$
- $(a_\infty \star \phi)(\boldsymbol{u}) = \int_{\mathbb{R}^\nu} a_\infty(\boldsymbol{t}) \phi(\boldsymbol{t}+\boldsymbol{u}) \mathrm{d}\boldsymbol{t}$: (usual) convolution,
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Theorem (3)

Let X be a linear RF satisfying Assumption (A)$(d; \mathbb{Z}^\nu)$/(A)$(d; \mathbb{R}^\nu)$. Then

$$\lambda^{-(\nu+4d)/2} X_\lambda(\phi) \xrightarrow{\mathrm{d}} \begin{cases} W_d(\phi), & 0 < d < \nu/4,\ \phi \in \Phi, \\ W_d(\phi), & -\nu/4 < d < 0,\ \phi \in \Phi_d^-, \\ \sigma W_0(\phi), & d = 0,\ \phi \in \Phi, \end{cases}$$

where $\sigma := \sum_{\boldsymbol{t} \in \mathbb{Z}^\nu} a(\boldsymbol{t}) / \int_{\boldsymbol{t} \in \mathbb{R}^\nu} a(\boldsymbol{t}) \mathrm{d}\boldsymbol{t}$.

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Let $A \subset \mathbb{R}^\nu$ be a bounded Borel set and $X = \{X(\boldsymbol{t}); \boldsymbol{t} \in \mathbb{R}^\nu\}$ be a stationary RF. Then

$$F_\lambda(y) := \frac{\int_{\lambda A} \mathbb{I}(X(\boldsymbol{t}) \le y)\mathrm{d}\boldsymbol{t}}{\mathrm{Leb}_\nu(\lambda A)}, \quad y \in \mathbb{R} \tag{20}$$

is the *empirical process* (empirical d.f.) of the marginal d.f. $F(y) = \mathrm{P}(X(\boldsymbol{t}) \le y)$ from observations on a large 'inflated' set $\lambda A, \lambda \to \infty$

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5. Nonlinear functionals and empirical processes

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- If G has Hermite rank 1: $h_1 := \mathrm{E}G(X(\boldsymbol{t}))X(\boldsymbol{t}) \ne 0$ the limit of $Y_\lambda(\phi)$ coincides with that of $h_1 X_\lambda(\phi)$ which is Gaussian

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- This talk: a similar result for nongaussian linear RX X with h_1 replaced by $a_1 =$ the first Appell coefficient of G

5. Nonlinear functionals and empirical processes

In Thm 4 X is a linear LRD RF on $\mathbb{Z}^\nu$:

$$X(\boldsymbol{t}) = \sum_{\boldsymbol{s} \in \mathbb{Z}^\nu} a(\boldsymbol{t} - \boldsymbol{s})\varepsilon(\boldsymbol{s}), \quad \boldsymbol{t} \in \mathbb{Z}^\nu,$$

with MA coefficients $a(\boldsymbol{t})$ satisfying Assumption $(\mathrm{A})(d; \mathbb{Z}^\nu), 0 < d < \nu/4$, and i.i.d. zero mean innovations satisfying moment and regularity conditions:

$$\mathrm{E}|\varepsilon|^{2p} < \infty \quad (\exists p \geq 2, p \in \mathbb{N}), \tag{22}$$

$$|\mathrm{E}\mathrm{e}^{\mathrm{i}z\varepsilon}| \leq C/(1 + |z|)^\tau, \ z \in \mathbb{R}, \quad (\exists\ C, \tau > 0). \tag{23}$$

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Let X be as above, and $G : \mathbb{R}\to\mathbb{R}$ be a measurable function satisfying

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Then X has infinitely differentiable marginal density $f(x), x\in\mathbb{R}$ and the first Appell coefficient of G

$$a_1 := -\int_{\mathbb{R}} G(x) f'(x)\mathrm{d}x$$

is well-defined.

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where $W_d(\phi)$ is Gaussian RF (the same Gaussian RF as in Thm 3) with zero mean and variance

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Set

$$\sigma_A^2 := \int_{\mathbb{R}^\nu} \Big(\int_A a_\infty(\boldsymbol{t} - \boldsymbol{s})\mathrm{d}\boldsymbol{t}\Big)^2 \mathrm{d}\boldsymbol{s}.$$

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In particular, K-S statistic $\mathcal{D}_\lambda = \sup_{y\in\mathbb{R}} |F_\lambda(y) - F(y)|$ satisfies

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- Tightness following Dehling & Taqqu (1989) chaining argument

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Let X satisfy Thm 3 with $p=2$, and $A \subset \mathbb{R}^\nu$ be a bounded Borel set. Then

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- Related/similar results obtained for nonlinear functions and empirical process of continuous argument LRD RF

$$X(\boldsymbol{t}) = \int_{\mathbb{R}^\nu} a(\boldsymbol{t} - \boldsymbol{s}) M(\mathrm{d}\boldsymbol{s}), \quad \boldsymbol{t} \in \mathbb{R}^\nu,$$

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- In causal LRD time series case ($\nu = 1$), (27) is shown by telescoping $G(X(t))$ onto orthogonal subspaces generated by lagged innovations (Ho & Hsing (1996, 1997), ...)

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M estimation in spatial linear regression

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Linear regression model:

$$Y_\lambda(\boldsymbol{t}) = \langle \beta, \boldsymbol{v}_\lambda(\boldsymbol{t}) \rangle + X(\boldsymbol{t}), \quad \boldsymbol{t} \in \lambda A \tag{30}$$

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- Regressors $v_{i,\lambda}(\boldsymbol{t}) = v_i(\boldsymbol{t}/\lambda)$ with non-degenerate $q \times q$ 'design matrix' $\mathbf{V} := \big(\int_A v_i(\boldsymbol{t}) v_j(\boldsymbol{t}) \mathrm{d}\boldsymbol{t}\big)_{i,j=1,\cdots,q}$

- Following Thm 3, the LS estimator

$$\widetilde{\beta}_{\lambda,\mathrm{LS}} = \mathbf{V}_\lambda^{-1} \int_{\lambda A} \boldsymbol{v}_\lambda(\boldsymbol{t}) Y_\lambda(\boldsymbol{t}) \mathrm{d}\boldsymbol{t}$$

is asymptotically normal:

$$\lambda^{(\nu-4d)/2}(\widetilde{\beta}_{\lambda,\mathrm{LS}} - \beta) \xrightarrow{\mathrm{d}} \boldsymbol{W}_{d,\boldsymbol{v}}(A) := \int_{\mathbb{R}^\nu} \Big\{ \int_A \mathbf{V}^{-1} \boldsymbol{v}(\boldsymbol{t}) a_\infty(\boldsymbol{t}-\boldsymbol{u}) \mathrm{d}\boldsymbol{t} \Big\} W(\mathrm{d}\boldsymbol{u})$$

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- Since LS $\widetilde{\beta}_{\lambda,\mathrm{LS}}$ is sensitive to outliers, a class of robust M estimators is considered where residuals $Y_\lambda(\boldsymbol{t}) - \langle \boldsymbol{z}, \boldsymbol{v}_\lambda(\boldsymbol{t}) \rangle$ are discounted by a nonlinear score function $\tau(y), y \in \mathbb{R}$ with $|\tau(y)| = o(|y|), |y| \to \infty$

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Theorem (5)

Consider the linear regression model in (30) *with regressor function* $\boldsymbol{v}_\lambda(\boldsymbol{t}) = \boldsymbol{v}(\boldsymbol{t}/\lambda)$, $\boldsymbol{v}(\cdot) \in L^1(\mathbb{R}^\nu) \cap L^\infty(\mathbb{R}^\nu)$ *and errors* X *being a linear LRD RF as in Corollary 1. Then for any score function* $\tau : \mathbb{R} \to \mathbb{R}$ *satisfying the above conditions, M estimator is asymptotically equivalent to LS estimator:*

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- Proof of Thm 5 follows the approach in Koul, (2002) *Weighted Empirical Processes in Dynamic Nonlinear Models.* (2002, Springer)

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Consider the linear regression model in (30) *with regressor function* $\boldsymbol{v}_\lambda(\boldsymbol{t}) = \boldsymbol{v}(\boldsymbol{t}/\lambda)$, $\boldsymbol{v}(\cdot) \in L^1(\mathbb{R}^\nu) \cap L^\infty(\mathbb{R}^\nu)$ *and errors* X *being a linear LRD RF as in Corollary 1. Then for any score function* $\tau : \mathbb{R} \to \mathbb{R}$ *satisfying the above conditions, M estimator is asymptotically equivalent to LS estimator:*

$$\lambda^{(\nu-4d)/2}(\widetilde{\beta}_{\lambda,\mathrm{M}} - \beta) = \lambda^{(\nu-4d)/2}(\widetilde{\beta}_{\lambda,\mathrm{LS}} - \beta) + o_p(1)$$

and has the same Gaussian limit distribution.

- Related results about equivalence of M and LS estimators in LRD time series, $\nu = 1$: Beran (1992), Koul & Mukherjee (1993), Wu (2003), ...
- Proof of Thm 5 follows the approach in Koul, (2002) *Weighted Empirical Processes in Dynamic Nonlinear Models.* (2002, Springer)
- uniform reduction principle for weighted empirical process