A log-linear model for non-stationary time series of counts

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1. GARCH and integer-valued GARCH
2. Models for non-stationary count processes
3. Mixing properties of INGARCH processes

GARCH and integer-valued GARCH

(Classical) GARCH:

$$\begin{aligned} X_t &= \sigma_t \, \varepsilon_t, \\ \sigma_t^2 &= f(X_{t-1}^2, \ldots, X_{t-p}^2; \sigma_{t-1}^2, \ldots, \sigma_{t-q}^2), \end{aligned}$$

where $f: \ \mathbb{R}_+^{p+q} \to \mathbb{R}_+$, $(\varepsilon_t)_{t\in\mathbb{Z}}$ sequence of i.i.d. rv's, $\mathbb{E}\varepsilon_t^2 = 1$.

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Poisson-INGARCH: $Q(\sigma) = \text{Poi}(\sigma) \quad \rightsquigarrow \quad \sigma_t = \text{var}(X_t \mid X_{t-1}, X_{t-2}, \ldots)$

A model for count series with a strong trend

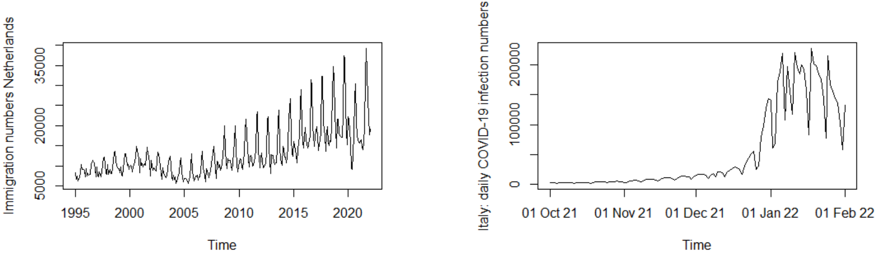

Figure: left: Monthly immigration numbers for the Netherlands with increasing trend and strongly increasing seasonality; right: daily COVID-19 infection numbers from Italy with explosive trend.

Poisson-INGARCH and trend

$$X_t \mid \text{``past''} \sim \text{Poi}(\sigma_t); \qquad \sigma_{t+1} = a_t X_t + b_t \sigma_t + Z_t,$$

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- if $\sqrt{\mathrm{var}(X_t \mid \sigma_t)}/E(X_t \mid \sigma_t) \stackrel{P}{\to} 0$, then
  - $d_{TV}(\text{Poi}(\widetilde{\sigma}_t), \text{Poi}(\widetilde{\sigma}'_t)) \stackrel{P}{\longrightarrow} 1$,
  - mixing properties deteriorate as $t \to \infty$

A remedy: (nearly) scale-invariant distributions on $\mathbb{N}_0$

Classical GARCH:

$$X_t \mid \text{"past"} \sim \mathcal{N}(0, \sigma_t)$$

$$\rightsquigarrow \qquad E\big(|X_t| \,\big|\, \text{"past"}\big) = \sqrt{2/\pi}\,\sqrt{\mathrm{var}(X_t \mid \text{"past"})}.$$

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Integer-valued counterparts:

$$X_\sigma = \lfloor \sigma Y \rfloor$$

$$(X_\sigma = k \qquad \Longleftrightarrow \qquad \sigma Y \in [k, k+1))$$

Examples

- $Y \sim \text{Exp}(1)$, then

$$P\big(X_\sigma = k\big) \;=\; P\big(\sigma Y \in [k, k+1)\big) \;=\; p\,(1-p)^k,$$

$p = 1 - e^{-1/\sigma}$

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- $Z \sim \mathcal{N}(0, \pi/2)$, then $Y := |Z|$ has a half-normal distribution, $EY = 1$, $X_\sigma \;=\; \lfloor \sigma Y \rfloor$

The proposed model

$$X_t \mid \text{“past”} \stackrel{d}{=} \lfloor \sigma_t Y \rfloor, \tag{2.1a}$$

$$\sigma_t = f\left(\sigma_{t-1}, X_{t-1}\right) \cdot Z_{t-1}, \tag{2.1b}$$

$(Z_t)_t$ sequence of exogenous covariates

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Then

$$d_{TV}\left(P^{X_\sigma}, P^{X_{\sigma'}}\right) \leq const. \cdot \left| \ln(\sigma) - \ln(\sigma') \right| \qquad \forall \sigma, \sigma' > 0$$

$\Longrightarrow$ change over to logarithmic scale

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(2.1b) equivalent to

$$\ln(\sigma_t) = \ln\left(f(\sigma_{t-1}, X_{t-1})\right) + \underbrace{\ln(Z_{t-1})}_{=: C_{t-1}} \tag{2.1c}$$

Covid data

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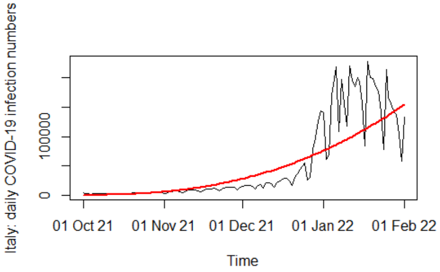
Least squares fit: $\widehat{\theta}_n = 2.48, \quad t \mapsto t^{\widehat{\theta}_n}$



Mixing vs. ergodicity

Applications in statistics: consistency, asymptotic normality, ...

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- (strictly) stationary processes
  ergodicity (plus some structure) suffices: ergodic theorem, CLT for martingale differences
- non-stationary processes
  ergodicity does not help, mixing (or "weak dependence")

Absolute regularity (β-mixing)

- $(\Omega, \mathcal{F}, P)$ probab. space, $\mathcal{A}$, $\mathcal{B}$ sub-σ-algebras of $\mathcal{F}$. Then

$$\beta(\mathcal{A}, \mathcal{B}) = \mathbb{E}\left[\sup_{B \in \mathcal{B}} |P(B \mid \mathcal{A}) - P(B)|\right].$$

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- $Y = (Y_t)_{t \in \mathbb{Z}}$ stochastic process on $(\Omega, \mathcal{F}, P)$. Then

$$\beta^Y(k, n) = \beta\left(\sigma(Y_t:\ t \leq k), \sigma(Y_t:\ t \geq k+n)\right),$$

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$$\begin{aligned} \beta^Y(k,n) &= \beta\left(\sigma(Y_t\colon\ t \le k), \sigma(Y_t\colon\ t \ge k+n)\right), \\ \beta^Y(n) &= \sup_k \beta^Y(k,n) \\ &= \sup_k \mathbb{E}\Big[\sup_C \big|P((Y_{k+n}, Y_{k+n+1}, \ldots) \in C \mid Y_k, Y_{k-1}, \ldots) \\ &\qquad - P((Y_{k+n}, Y_{k+n+1}, \ldots) \in C)\big|\Big] \end{aligned}$$

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\end{aligned}$$

Y is absolutely regular (β-mixing) if

$$\beta^Y(n) \underset{n \to \infty}{\longrightarrow} 0.$$

Mixing of classical GARCH

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Method of proof:

- $\big((X_t, \sigma_t)\big)_t$ time-homogeneous Markov chain
- MC technology can be used

Here: Mixing of INGARCH(1,1)

- $\big((X_t, \sigma_t)\big)_t$ is a Markov chain
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Consequences:

- Cannot use MC technology; use coupling arguments instead
- Revision of (a possible) original goal: Prove mixing only for the count process $(X_t)_t$

Non-mixing of $(\sigma_t)_t$

Counterexample (Neumann, 2011, *Bernoulli*):

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Then $\quad 2\sigma_t = X_{t-1} + \underbrace{2g(\sigma_{t-1})}_{\in (0,1)} \quad \rightsquigarrow \quad \lfloor 2\sigma_t \rfloor = X_{t-1}$

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- g strictly monotone $\quad \rightsquigarrow \quad$ we can recover σ_{t-1} from σ_t
  $\rightsquigarrow \quad$ we can recover from σ_t the complete past, $(\sigma_s)_{s<t}$
- $(\sigma_t)_t$ is not purely non-random
  $\rightsquigarrow \quad$ not strong (α-) mixing $\quad \rightsquigarrow \quad$ not β-mixing

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&= \widetilde{P}\left(\widetilde{X}_{k+n} \neq \widetilde{X}'_{k+n} \right) \\
&\quad + \sum_{r=1}^{\infty} \widetilde{P}\left(\widetilde{X}_{k+n+r} \neq \widetilde{X}'_{k+n+r}, \widetilde{X}_{k+n+r-1} = \widetilde{X}'_{k+n+r-1}, \ldots, \widetilde{X}_{k+n} = \widetilde{X}'_{k+n} \right).
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Mixing of "Scale-invariant" INGARCH

$$X_t \mid \text{``past''} \stackrel{d}{=} \lfloor \sigma_t Y \rfloor, \tag{3.1a}$$

$$\ln(\sigma_t) = \ln\big(f(\sigma_{t-1}, X_{t-1})\big) + \underbrace{\ln(Z_{t-1})}_{=:\, C_{t-1}} \tag{3.1b}$$

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Contractive condition:

$$\big|\ln\big(f(x,\sigma)\big) - \ln\big(f(x',\sigma')\big)\big| \leq a\,\big|\ln(\sigma) - \ln(\sigma')\big| + b\,\big|\ln(x+1) - \ln(x'+1)\big|.$$

Mixing of "Scale-invariant" INGARCH

$$X_t \mid \text{"past"} \stackrel{d}{=} \lfloor \sigma_t Y \rfloor, \qquad (3.1a)$$

$$\ln(\sigma_t) = \ln\big(f(\sigma_{t-1}, X_{t-1})\big) + \underbrace{\ln(Z_{t-1})}_{=:\, C_{t-1}} \qquad (3.1b)$$

Contractive condition:

$$\Big|\ln\big(f(x,\sigma)\big) - \ln\big(f(x',\sigma')\big)\Big| \leq a \,\big|\ln(\sigma) - \ln(\sigma')\big| + b\,\big|\ln(x+1) - \ln(x'+1)\big|.$$

Theorem 3.1 (Leucht & N., 2023+).

If $a + \gamma b < 1$, $\sup\{E|C_t - EC_t| \colon\ t \in \mathbb{N}_0\} < \infty$, *and* $E|\ln(\sigma_0)| < \infty$, *then*

$$\beta^X(n) = O\big(\rho^n\big)$$

for some $\rho < 1$.

Proof of Theorem 3.1

For $t \geq k$, couple $\widetilde{X}_t$ and $\widetilde{X}'_t$ such that

$$\begin{aligned}
\widetilde{P}\big(\widetilde{X}_t \neq \widetilde{X}'_t \mid \widetilde{\sigma}_t, \widetilde{\sigma}'_t\big) &= d_{TV}\big(P^{\lfloor \widetilde{\sigma}_t Y \rfloor}, P^{\lfloor \widetilde{\sigma}'_t Y \rfloor}\big) \\
&= O\big(|\ln(\widetilde{\sigma}_t) - \ln(\widetilde{\sigma}'_t)|
\end{aligned}$$

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Then

$$\begin{aligned} &\beta^X(k,n) \\ &\quad \leq \underbrace{\widetilde{P}\left(\widetilde{X}_{k+n} \neq \widetilde{X}_{k+n}'\right)}_{\leq c_1(a+\gamma b)^n} \\ &\qquad + \sum_{r=1}^{\infty} \underbrace{\widetilde{P}\left(\widetilde{X}_{k+n+r} \neq \widetilde{X}_{k+n+r}', \widetilde{X}_{k+n+r-1} = \widetilde{X}_{k+n+r-1}', \ldots, \widetilde{X}_{k+n} = \widetilde{X}_{k+n}'\right)}_{\leq c_2(a+\gamma b)^n a^r}. \end{aligned}$$

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- INGARCH: proof of mixing via coupling rather than existing MC results
- Poisson-INGARCH: a moderate trend leaves mixing properties intact
- "scale-invariant" INGARCH:
  - closer to classical GARCH
  - contraction at a logarithmic scale $\rightsquigarrow$ explosive behavior does not affect mixing properties