Self-normalized sums of heavy-tailed time series

Thomas Mikosch (University of Copenhagen), Olivier Wintenberger (Sorbonne, Pauli-Institut), Munyea Matsui (Nagoya)

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Figure 1. Friendly yours, Paul

Extremes and Local Dependence in Stationary Sequences

M.R. Leadbetter*

University of North Carolina, Dept. of Statistics, 321 Phillips Hall 039A, Chapel Hill, N.C. 27514, USA

Summary. Extensions of classical extreme value theory to apply to stationary sequences generally make use of two types of dependence restriction:

(a) a weak "mixing condition" restricting long range dependence

(b) a local condition restricting the "clustering" of high level exceedances.

The purpose of this paper is to investigate extremal properties when the local condition (b) is omitted. It is found that, under general conditions, the type of the limiting distribution for maxima is unaltered. The precise modifications and the degree of clustering of high level exceedances are found to be largely described by a parameter here called the "extremal index" of the sequence.

1. Introduction

Classical Extreme Value Theory discusses the possible limiting laws for the maximum

$$M_n = \max(\xi_1, \xi_2 \dots \xi_n) \qquad (1.1)$$

1. The Leadbetter conditions D and D' (1974,1983)

- **Consider an iid sequence (X_t) with common distribution F and partial maxima**

$$M_n = \max_{i=1,\dots,n} X_i\,, \qquad n \ge 1\,.$$

- For a threshold sequence $u_n(\tau) \to x_F$, $\tau \in [0,\infty]$,

$$\mathbb{P}(M_n \le u_n(\tau)) \to \mathrm{e}^{-\tau}\,, \qquad n \to \infty\,,$$

holds if and only if

$$n\,\overline{F}(u_n(\tau)) \to \tau\,, \qquad n \to \infty\,.$$

- In particular, for $u_n(x) = c_n\,x + d_n$, $c_n > 0$, $d_n \in \mathbb{R}$,

$$\mathbb{P}\big(c_n^{-1}(M_n - d_n) \le x\big) \to H(x)\,, \qquad x \in \mathbb{R}\,,$$

for an extreme value distribution H if and only if

$$n\,\overline{F}(c_n\,x + d_n) \to -\log H(x)\,.$$

What happens if (X_t) is (strictly) stationary?

- Condition $D(u_n(\tau))$ is a *mixing condition* motivated by the *blocks method* for $r_n/n \to 0$, $k_n = [n/r_n] \to \infty$.

$$\underbrace{X_1,\ldots,X_{r_n}}_{\text{Block } 1},\underbrace{X_{r_n+1},\ldots,X_{2\,r_n}}_{\text{Block } 2},\ldots,\underbrace{X_{(k_n-1)\,r_n+1},\ldots,X_{k_n\,r_n}}_{\text{Block } k_n}\,.$$

- **A stronger version of $D(u_n(\tau))$**

$$
\begin{aligned}
&\mathbb{P}\big(M_n \le u_n(\tau)\big)\\
&= \Big(\mathbb{P}\big(M_{r_n} \le u_n(\tau)\big)\Big)^{k_n} + o(1)\\
&= \exp\Big(-k_n\,\mathbb{P}\big(M_{r_n} > u_n(\tau)\big)(1+o(1))\Big) + o(1)\\
&= \exp\Big(-\underbrace{\frac{\mathbb{P}\big(M_{r_n} > u_n(\tau)\big)}{r_n\,\overline{F}(u_n(\tau))}}_{=:\theta_n(\tau)}\underbrace{\big[n\,\overline{F}(u_n(\tau))\big]}_{\to\tau}(1+o(1))\Big) + o(1)\\
&\to \exp\big(-\theta_n(\tau)\,\tau\,(1+o(1))\big)\quad n\to\infty\,.
\end{aligned}
$$

- If $\theta_n(\tau) \to \theta_X \in [0,1]$ this limit is the extremal index.
- $\theta_n(\tau)$ is the reciprocal of the expected number of exceedances of $u_n(\tau)$ in a block.
- $\theta_n(\tau) \to \theta_X$ is a large deviation result for maxima:

$$\mathbb{P}\big(M_{r_n} > u_n(\tau)\big) \sim \theta_X \, r_n \, \overline{F}(u_n(\tau))$$

When is $\theta_X = 1$? As if (X_t) were iid...

- Leadbetter's anti-clustering condition

$$\lim_{k\to\infty} \limsup_{n\to\infty} \sum_{i=1}^{[n/k]} \mathbb{P}\big(X_i > u_n(\tau) \mid X_0 > u_n(\tau)\big) = 0 \,.$$

- **Satisfied for any reasonable Gaussian stationary sequence.**

2. Regular variation of stationary sequences

- **A strictly stationary sequence (X_t) is regularly varying with index $\alpha > 0$ if $|X_0|$ is regularly varying with index α and for every $h \geq 0$,** Davis, Hsing (1995)

$$\frac{\mathbb{P}\big(x^{-1}(X_0, \ldots, X_h) \in \cdot\big)}{\mathbb{P}(|X_0| > x)} \xrightarrow{v} \mu_h(\cdot)\,, \qquad x \to \infty\,.$$

- **A strictly stationary sequence (X_t) is regularly varying with index $\alpha > 0$ if there exists a sequence (Θ_t) independent of a Pareto(α)-distributed Y_α such that for every $h \geq 0$,**

$$\mathbb{P}\big(x^{-1}(X_{-h}, \ldots, X_h) \in \bullet \big| |X_0| > x\big) \xrightarrow{w} \mathbb{P}\big(Y_\alpha\,(\Theta_{-h}, \ldots, \Theta_h) \in \bullet\big)$$

- **(Θ_t) is the spectral tail process.** Basrak, Segers (2009)

2.1. Examples of regularly varying time series.

AR(1) process: $X_t = \varphi X_{t-1} + Z_t$, (Z_t) iid regularly varying with index $\alpha > 0$, $|\varphi| < 1$. Then (X_t) is regularly varying with index α and

$$\Theta_t = \Theta_0 \varphi^t , \quad t \geq 0 .$$

Affine stochastic recurrence equation: $X_t = A_t X_{t-1} + B_t$, (A_t, B_t), $t \in \mathbb{Z}$, iid, and the equation $\mathbb{E}[|A|^\alpha] = 1$ has a positive solution OR (B_t) is regularly varying with index α and $\mathbb{E}[|A|^\alpha] < 1$. Then (X_t) is regularly varying with index $\alpha > 0$ and

$$\Theta_t = \Theta_0 A_1 \cdots A_t \quad t \geq 0 .$$

Kesten (1973), Goldie (1991), Grincevičius (1985)

GARCH(1, 1) process: $X_t = \sigma_t Z_t$, (Z_t) iid, $\mathbb{E}[Z] = 0$, $\mathbb{E}[Z^2] = 1$,

$$\sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \beta_1 \sigma_{t-1}^2 = \alpha_0 + (\alpha_1 Z_{t-1}^2 + \beta_1) \sigma_{t-1}^2 .$$

(σ_t^2) satisfies an affine stochastic recurrence equation.

It is regularly varying with index $\alpha/2$ if $\mathbb{E}[(\alpha_1 Z_0^2 + \beta_1)^{\alpha/2}] = 1$ and (X_t) inherits regular variation with index α.

Stochastic volatility model: $X_t = \sigma_t Z_t$, (σ_t) positive stationary, independent of an iid regularly varying sequence (Z_t) with index α. If $\mathbb{E}[\sigma^{\alpha+\delta}] < \infty$ for some $\delta > 0$, (X_t) is regularly varying with index α and $\Theta_t = 0$, $t \neq 0$.

Asymptotic independence

2.2. Limit theory for partial maxima.

Here we assume that (X_t) is a non-negative stationary regularly varying sequence with index $\alpha > 0$, and (a_n) satisfies $n\,\mathbb{P}(X_0 > a_n) \to 1$.

Mixing condition

$$\mathbb{P}(M_n \le x\,a_n) - \big[\mathbb{P}(M_{r_n} \le x\,a_n)\big]^{k_n} \to 0\,, \qquad n \to \infty\,,$$

Anti-clustering condition:

$$\lim_{k\to\infty} \limsup_{n\to\infty} \mathbb{P}\big(M_{k,r_n} > x\,a_n \mid X_0 > x\,a_n\big) = 0\,, \quad x > 0\,.$$

Then a telescoping sum argument Jakubowski (1993,1997)

$\mathbb{P}(M_k > u) - \mathbb{P}(M_{k-1} > u) = \mathbb{P}(X_0 > u\,, M_{k-1} \le u)$ **shows**

$$\lim_{k\to\infty} \limsup_{n\to\infty} \Big| \underbrace{\frac{\mathbb{P}(M_{r_n} > x\,a_n)}{r_n\,\mathbb{P}(X_0 > x\,a_n)}}_{=:\theta_n(x)} - \mathbb{P}(M_k \le x\,a_n \mid X_0 > x\,a_n)\Big| = 0\,.$$

- **By regular variation and the continuous mapping theorem**

$$\mathbb{P}(M_k \le x\, a_n \mid X_0 > x\, a_n) \overset{n\to\infty}{\to} \mathbb{P}\big(\max_{i=1,\dots,k} Y_\alpha\, \Theta_i \le 1\big)$$
$$= \mathbb{P}\big(Y_\alpha^\alpha \max_{i=1,\dots,k} \Theta_i^\alpha \le 1\big)$$
$$\overset{k\to\infty}{\to} \mathbb{P}\big(Y_\alpha^\alpha \sup_{i=1,2,\dots} \Theta_i^\alpha \le 1\big) = \theta_X$$

- **AR(1) process ($|X_t|$):**

$$\mathbb{P}\big(\sup_{i=1,2,\dots} |\varphi|^{i\alpha} \le Y_\alpha^{-\alpha}\big) = 1 - |\varphi|^\alpha = \theta_{|X|}$$

- **Stoch. recurrence eqn.: $A, B \ge 0$ a.s.**

$$\mathbb{P}\big(\sup_{i=1,2,\dots} A_1^\alpha \cdots A_i^\alpha \le Y_\alpha^{-\alpha}\big) = \mathbb{E}\Big[\big(1 - \sup_{i=1,2,\dots} A_1^\alpha \cdots A_i^\alpha\big)_+\Big] = \theta_X$$

- **Stochastic volatility model: $\Theta_t = 0$, $t = 1, 2, \dots$: $\theta_X = 1$.**

- **Under mixing and anti-clustering,**
  - $*$ **for the time T^* of the largest record of $(|\Theta_t|)_{t\in\mathbb{Z}}$,**

$$\theta_{|X|} = \mathbb{P}(T^* = 0) = \mathbb{P}\Big(\sup_{t\le -1} |\Theta_t| < 1 = \sup_{t\ge 0} |\Theta_t|\Big)$$

  - $*$ $\Theta_t \to 0$, $|t| \to \infty$ **and** $\sum_{t\in\mathbb{Z}} |\Theta_t|^\alpha < \infty$.

3. α-Stable limit theory for partial sums, $\alpha \in (0,2)$

- (X_t) stationary regularly varying with index α, generic element X, normalizing constants (a_n) with $n\,\mathbb{P}(|X_0| > a_n) \to 1$, and partial sums

$$S_n = X_1 + \cdots + X_n\,, \qquad n \ge 1\,.$$

Mixing condition: The characteristic functions of $a_n^{-1} S_n$ and $a_n^{-1} S_{r_n}$ satisfy

$$\varphi_{a_n^{-1} S_n}(u) = \big(\varphi_{a_n^{-1} S_{r_n}}(u)\big)^{k_n} + o(1)\,, \qquad n \to \infty\,, \quad u \in \mathbb{R}\,.$$

Anti-clustering condition:

$$\lim_{k\to\infty} \limsup_{n\to\infty} n \sum_{j=k}^{r_n} \mathbb{E}\big[(|a_n^{-1}\mathrm{X}_j| \wedge 1)\,(|a_n^{-1}\mathrm{X}_0| \wedge 1)\big] = 0\,.$$

- **Then $a_n^{-1}(S_n - b_n) \xrightarrow{d} \xi_\alpha$ and ξ_α is α-stable with characteristic function ($\alpha \neq 1$)**

$$\varphi_{\xi_\alpha}(u) = \exp\big(-c_\alpha\,\sigma^\alpha(u)\big(1 - i\,\beta(u)\tan(\alpha\,\pi/2)\big)\big)\,, \qquad u \in \mathbb{R}\,.$$

with

$$\begin{aligned}\beta(u) &= \frac{\mathbb{E}\Big[\big((u\sum_{i=0}^\infty \Theta_i)_+^\alpha - (u\sum_{i=1}^\infty \Theta_i)_+^\alpha\big) - \big((u\sum_{i=0}^\infty \Theta_i)_-^\alpha - (u\sum_{i=1}^\infty \Theta_i)_-^\alpha\big)\Big]}{\mathbb{E}[|u\sum_{i=0}^\infty \Theta_i|^\alpha - |u\sum_{i=1}^\infty \Theta_i|^\alpha]}\,,\\ \sigma^\alpha(\mathrm{u}) &= \mathbb{E}\Big[\Big|u\sum_{i=0}^\infty \Theta_i\Big|^\alpha - \Big|u\sum_{i=1}^\infty \Theta_i\Big|^\alpha\Big]\,,\end{aligned}$$

- Assume $b_n = 0$. Under mixing, $a_n^{-1} S_n \xrightarrow{d} \xi_\alpha$ if and only if $a_n^{-1} \sum_{i=1}^{k_n} S'_{r_n,i} \xrightarrow{d} \xi_\alpha$ for iid copies $S'_{r_n,i}$ of S_{r_n}. Then Petrov (1974)

$$k_n \, \mathbb{P}(\pm S_{r_n} > x \, a_n) \sim \frac{\mathbb{P}(\pm S_{r_n} > x \, a_n)}{r_n \, \mathbb{P}(|X_0| > a_n)} \to c_\pm \, x^{-\alpha}, \quad x > 0 \, .$$

- This is a large deviation result.

- It extends to uniform convergence.

Extremal indices for sums.

- **Linear process.** $X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$, (Z_t) **iid regularly varying,** $\alpha \in (0,2)$. **If all non-zero** ψ_j **have the same sign or** Z **is symmetric, then**

$$\varphi_{\xi_\alpha}(u) = \big(\varphi_{\xi'_\alpha}(u)\big)^{|\sum_j \psi_j|^\alpha / \sum_j |\psi_j|^\alpha}$$

  ξ'_a **is the limit for sums of iid copies of** X_t.

- **Affine SRE.** $X_t = A_t X_{t-1} + B_t$, $A, B \geq 0$.

$$\varphi_{\xi_\alpha}(u) = \big(\varphi_{\xi'_\alpha}(u)\big)^{\mathbb{E}\left[\left(1+\sum_{j=1}^{\infty} A_1\cdots A_j\right)^\alpha - \left(\sum_{j=1}^{\infty} A_1\cdots A_j\right)^\alpha\right]}$$

Similar conditions and arguments apply to

- **point process convergence (using Laplace functionals) Davis, Hsing (1995), Basrak, Segers (2009),**
- **convergence of ℓ^p-norms of samples (using Laplace transforms),**
- **their convergence jointly with maxima and sums (hybrid characteristic functions)**

4. Maxima, sums, and their ratios

- **Under anti-clustering for sums and mixing for the hybrid characteristic function**

$$\mathbb{E}\big[\mathrm{e}^{i\,u a_n^{-1}(S_n-b_n)}\mathbf{1}(a_n^{-1}M_n^{|X|}\le x)\big]\,,\quad u\in\mathbb{R},\ x>0\,,$$

we have for $\alpha\in(0,2)$,

$$a_n^{-1}(M_n^{|X|},S_n-b_n)\stackrel{d}{\rightarrow}(\eta_\alpha,\xi_a)$$

where

$$\begin{aligned}
&\mathbb{E}\big[\mathrm{e}^{i\,u\xi_\alpha}\mathbf{1}(\eta_\alpha\le x)\big]\\
&= \varphi_{\xi_\alpha}(u)\,\exp\Big(-\int_0^\infty \mathbb{E}\Big[\mathrm{e}^{i\,y\,u\sum_{t=-\infty}^{\infty}Q_t}\,\mathbf{1}\Big(y\max_{t\in\mathbb{Z}}|Q_t|>x\Big)\Big]\,d(-y^{-\alpha})\Big)\\
&= \varphi_{\xi_\alpha}(u)\,\Phi_\alpha^{\theta_{|X|}}(x)\,\exp\Big(-\theta_{|X|}\int_x^\infty \mathbb{E}\big[\mathrm{e}^{i\,y\,u\sum_{t=-\infty}^{\infty}\widetilde{Q}_t}-1\big]\,d(-y^{-\alpha})\Big)\\
&\qquad u\in\mathbb{R}\,,\qquad x>0\,,
\end{aligned}$$

and $(Q_t)=\big(\Theta_t/\big(\sum_{i\in\mathbb{Z}}|\Theta_i|^\alpha\big)^{1/\alpha}$ **and** $(\widetilde{Q}_t)$ **is a version of** (Q_t) **under some change of measure.**

- In the case of asymptotic independence: $\Theta_t = Q_t = \widetilde{Q}_t = 0$, $t \neq 0$, but ξ_α and η_α are not independent.
- Independence of η_α, ξ_α is only possible if $\sum_{t\in\mathbb{Z}} Q_t = \sum_{t\in\mathbb{Z}} \Theta_t = 0$ a.s. This implies $\xi_\alpha = 0$ a.s. For a linear process, this corresponds to $\sum_j \psi_j = 0$.
- Ratio limit

$$\frac{S_n - b_n}{M_n^{|X|}} \stackrel{d}{\to} R_\alpha = \frac{\xi_\alpha}{\eta_\alpha},$$

where for $u \in \mathbb{R}$

$$\varphi_{R_\alpha}(u) = \frac{\mathbb{E}\Big[e^{i\,u\sum_{t=-\infty}^{\infty}\widetilde{Q}_t}\Big]}{\int_0^\infty \mathbb{E}\Big[1 + i\,y\,u\sum_{t\in\mathbb{Z}}\widetilde{Q}_t\,1_{(1,2)}(\alpha) - e^{iyu\sum_{t=-\infty}^{\infty}\widetilde{Q}_t}1(y\le 1)\Big]\,d(-y^{-\alpha})}.$$

- **AR(1) process:** $X_t = \varphi X_{t-1} + Z_t$ **for an iid regularly varying sequence such that** $\mathbb{P}(Z > x) = x^{-\alpha}$ **for** $x > 1$, $\varphi \in (-1, 1)$ **and** $\alpha \in (0, 1) \cup (1, 2)$. **Then**

$$\mathbb{E}[R_\alpha] = \frac{1}{1-\alpha}\,\mathbb{E}\Big[\sum_{t \in \mathbb{Z}} \widetilde{Q}_t\Big] = \frac{1}{(1-\alpha)(1-\varphi)}\,.$$

5. Self-normalizations

- **Write for a stationary regularly varying sequence (X_t) with index $\alpha \in (0,2)$,**

$$\gamma_{n,p} = \Big(\sum_{t=1}^{n} |X_t|^p\Big)^{1/p}, \qquad p > 0\,.$$

Under anti-clustering and mixing, for $\alpha < p$,

$$(a_n^{-1} S_n, a_n^{-1} M_n^{|\mathrm{X}|}, a_n^{-p}\gamma_{n,p}^{p}) \stackrel{d}{\rightarrow} (\xi_\alpha, \eta_\alpha, \zeta_{\alpha,p}^{p})\,, \qquad n \to \infty\,,$$

where the joint limit distribution is described by

$$\begin{aligned} &\mathbb{E}\big[\mathrm{e}^{i\,u\,\xi_\alpha}\,1(\eta_\alpha \le x)\,\mathrm{e}^{-\lambda^p\,\zeta_{\alpha,p}^p}\big] \\ &= \exp\Big(\int_0^\infty \mathbb{E}\Big[\mathrm{e}^{i\,y\,u\,\sum_{t=-\infty}^{\infty} Q_t - y^p\lambda^p \sum_{t=-\infty}^{\infty} |Q_t|^p}\,1\Big(y\,\max_{t\in\mathbb{Z}} |Q_t| \le x\Big) \\ &\qquad -1 - i\,y\,u \sum_{t\in\mathbb{Z}} Q_t\,1_{(1,2)}(\alpha)\Big]\,d(-y^{-\alpha})\Big)\,. \end{aligned}$$

$\zeta_{\alpha,p}^{p}$ **has an α/p-stable distribution.**

- **Studentized sums**

$$\frac{S_n}{\gamma_{n,p}} \xrightarrow{d} \frac{\xi_\alpha}{\zeta_{\alpha,p}} =: \mathrm{R}_{\alpha,p}\,, \qquad n \to \infty\,.$$

and

$$\mathbb{E}[R_{\alpha,p}] = \frac{\Gamma((1-\alpha)/p)}{\Gamma(1/p)\Gamma(1-\alpha/p)} \mathbb{E}\Big[\frac{\|Q\|_p^\alpha}{\mathbb{E}[\|Q\|_p^\alpha]} \frac{\sum_{t=-\infty}^{\infty} Q_t}{\|Q\|_p}\Big]\,.$$

- **Greenwood statistics** for a positive regularly varying stationary time series (X_t) , $\alpha < p \wedge 1$,

$$T_{n,p} := \frac{X_1^p + \cdots + X_n^p}{(X_1 + \cdots + X_n)^p} \xrightarrow{d} \frac{\zeta_{\alpha,p}^p}{\xi_\alpha^p}\,,$$

where $\zeta_{\alpha,p}^p$ is α/p-stable and ξ_α is α-stable.

$$\mathbb{E}[T_{n,p}] \to \mathbb{E}\Big[\frac{\zeta_{\alpha,p}^p}{\xi_\alpha^p}\Big] = \frac{\Gamma(p-\alpha)}{\Gamma(p)\,\Gamma(1-\alpha)} \mathbb{E}\Big[\frac{\|Q\|_1^\alpha}{\mathbb{E}[\|Q\|_1^\alpha]} \frac{\|Q\|_p^p}{\|Q\|_1^p}\Big]\,.$$

6. Another view at these limit results: point process convergence

- **The distribution of a point process $N = \sum_i \varepsilon_{Y_i}$ with state space $E \subset \mathbb{R}$ is determined by its Laplace functional**

$$\Psi_N(f) = \mathbb{E}\big[\exp\big(-\int_E f\,dN\big)\big]\,, \qquad f \in C_K^+\,,$$

and $N_n \stackrel{d}{\to} N$ if and only if $\Psi_{N_n}(f) \to \Psi_N(f)$, $f \in C_K^+$.

- **Assume (X_t) is stationary regularly varying with index $\alpha > 0$. Set**

$$N_n = \sum_{t=1}^{n} \varepsilon_{X_t/a_n}\,, \qquad n \geq 1\,.$$

- Davis, Hsing (1995) **Under anti-clustering and mixing in terms of the Laplace functionals of (N_n),**

$$N_n \xrightarrow{d} N = \sum_{i=1}^{\infty} \sum_{j\in\mathbb{Z}} \varepsilon_{\Gamma_i^{-1/\alpha} Q_{ij}}\,, \qquad n\to\infty\,,$$

on $E = \mathbb{R}_0 = \mathbb{R}\backslash\{0\}$, where $0<\Gamma_1<\Gamma_2<\cdots$ are the points of a homogeneous Poisson process on $(0,\infty)$ with intensity $\theta_{|X|}>0$, $\sum_{j\in\mathbb{Z}}\varepsilon_{Q_{ij}}$ are iid cluster processes with $\sup_j|Q_{ij}|=1$ a.s.

- **(Joint) limit theory for $a_n^{-1}(S_n-b_n, M_n^{|X|})$ follows by the continuous mapping theorem, e.g.**

$$\begin{aligned}\mathbb{P}\big(0\le a_n^{-1}M_n^{|X|}\le x\big) &\to \mathbb{P}\Big(\sup_{i\ge1}\Gamma_i^{-1/\alpha}\sup_{j\in\mathbb{Z}}|Q_{ij}|\le x\Big)\\ &= \mathbb{P}\big(\Gamma_1^{-1/\alpha}\le x\big) = \Phi_\alpha^{\theta_{|X|}}(x)\,.\end{aligned}$$

- A similar argument applies for sums, $\alpha \in (0,2)$, by first summing the largest points

$$\xi_\alpha(\delta) = \sum_{i=1}^{\infty} \Gamma_i^{-1/\alpha} \sum_{j\in\mathbb{Z}} Q_{ij}\, 1(\Gamma_i^{-1/\alpha}|Q_{ij}| > \delta)$$

and then letting $\delta \downarrow 0$: one needs to check the vanishing-small-values condition for $\alpha \in (1,2)$: for $\gamma > 0$,

$$\lim_{\delta\downarrow 0} \limsup_{n\to\infty} \mathbb{P}\Big(\Big|a_n^{-1} \sum_{t=1}^{n} X_t\, 1(a_n^{-1}|X_t| \le \delta) - \mathbb{E}[\ldots]\Big| > \gamma\Big) = 0\,.$$

- This is difficult and, in general, it is also difficult to identify the parameters of the α-stable limit.


PARETO'S
80ESTATE20
OLD VINE ZINFANDEL
LODI, CALIFORNIA
PARETO PRINCIPLE, ALSO CALLED THE 80/20 RULE, STATES THAT 80%
FROM 20% OF THE EFFORT. THIS HOLDS TRUE IN THE WORLD OF
WINE CONSUMED BY 20% OF THE PEOPLE. WE APPLAUD YOUR