Dynamic modeling of abundance data in ecology

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Outlines

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I. Introduction

II. Modeling relative abundance

III. Modeling Absence/Presence of species

Ecological definitions

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One can also consider the **Shannon entropy**

$$\forall y = (y_1, \ldots, y_d) \in \mathcal{S}_{d-1}, \ I_S(y) = -\sum_{i=1}^{d} y_i \log(y_i).$$

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  - ➜ The impact of exogenous variables.

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We focus on three groups of species:

(a) Coleoptera

(b) Hymenoptera

(c) Diptera

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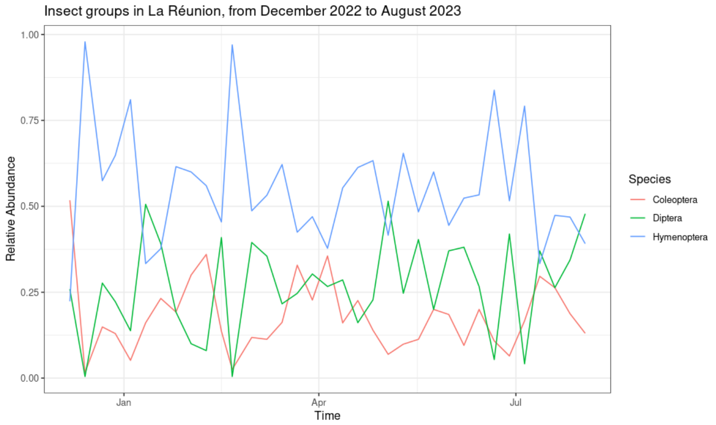

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where:

- P is a transition kernel from source $\mathcal{S}_{d-1}^{\mathbb{N}}$ and target $\mathcal{S}_{d-1}$,
- Y_t^- denotes the entire past of the time series at time t:

$$Y_t^- = \left(Y_t, Y_{t-1}, \ldots\right).$$

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  we obtain a Markov chain.
- It is possible to add a process of exogenous variables $(X_t)_{t\in\mathbb{Z}}$ to the dynamic of the process $(Y_t)_{t\in\mathbb{Z}}$

$$\mathbb{P}\left(Y_{t+1} \in A \mid Y_t^- = y_t^-, X_t^- = x_t^-\right) = P\left(A \mid y_t^-, x_t^-\right)$$

Existence of the process

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Theorem 1

Under assumptions **A1** and **A2** below, there exists a time series $(Y_t)_{t\in\mathbb{Z}}$ which is strictly stationary such that

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We have $\sum_{m\in\mathbb{N}} b_m < \infty$.

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- a dispersion parameter $\varphi > 0$.

Dirichlet model (2/3)

➜ In the spirit of the logistic regression, we propose that for all $t \in \mathbb{Z}$

$$\mathrm{alr}(\lambda_t) = \eta_0 + \sum_{k \geqslant 1} \eta_k \overline{Y}_{t-k} + \sum_{k \geqslant 1} \zeta_k X_{t-k},$$

where alr is the mapping

$$\begin{aligned} \mathrm{alr} : \quad & \mathcal{S}_{d-1} \longrightarrow \mathbb{R}^{d-1} \\ & y = (y_1, \dots, y_d) \longmapsto \left(\log\left(\frac{y_1}{y_d}\right), \dots, \log\left(\frac{y_{d-1}}{y_d}\right) \right), \end{aligned}$$

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➜ We also propose that for all $t \in \mathbb{Z}$

$$\varphi_t = \exp\left(\theta_0 + \sum_{k \geqslant 1} \theta_k I_S(Y_{t-k+1}) \right),$$

where the θ_k's are real numbers.

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- The θ's give us information about the volatility of the abundance. The idea is to connect the biodiversity of the ecosystem and its variability.

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An optimization of the conditional likelihood is performed to obtain an estimation of the parameters.

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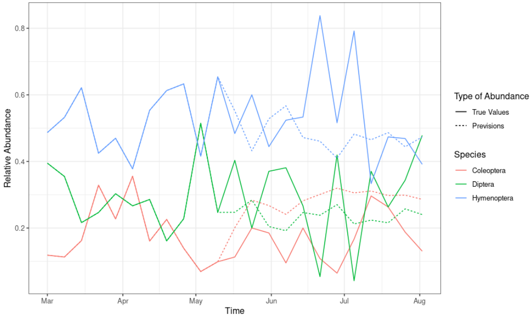

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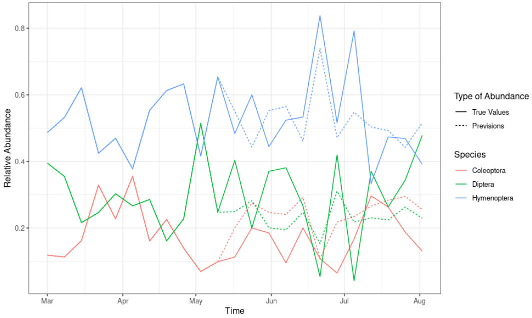

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- Model the absence/presence of species in the ecosystem.

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It is actually a dynamic version of a multivariate probit regression.

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In addition, its distribution is unique.

Remark

Furthermore, if $(\zeta_t)_{t\in\mathbb{Z}}$ is ergodic, $(Y_t)_{t\in\mathbb{Z}}$ is also ergodic.

Estimation results (1/3)

We consider first a single trajectory of an absence/presence process $(Y_t)_{1\leqslant t\leqslant T}$, and we are interested in the estimation of

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- We then maximize all pairwise conditional likelihoods

$$\widehat{R}(i,j) = \underset{r\ \in]-1,1[}{\operatorname{argmax}} \sum_{t=p+1}^{T} \log \left\{ \int_{I_{Y_{i,t}} - \hat{\lambda}_{i,t}} \Phi\left((2Y_{j,t} - 1) \frac{\hat{\lambda}_{j,t} + r x_i}{\sqrt{1 - r^2}} \right) \varphi(x_i) \mathrm{d}x_i \right\}$$

Estimation results (3/3)

Proposition 1

Assume the process ζ_t is ergodic. Under some reasonable assumptions on the covariates:

1) All estimators $\hat{\theta}$, $\hat{\gamma}$ and $\hat{R}$ are strongly consistent.

2) Moreover, we have the asymptotic normality of

$$\sqrt{T-p}\left(\hat{\theta}-\theta_0\right) \quad \text{and} \quad \sqrt{T-p}\left(\hat{\gamma}-\gamma_0, \widehat{R}-R_0\right).$$

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Generalize the results about consistency and central limit theorems for M-estimators.

Results about M-estimators (1/2)

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- and by adding the following "order condition"

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for some $\delta > 0$.

Estimation Results for panel data (1/2)

In the case of panel data, we can consider similar estimators as the ones mentioned previously

$$\hat{\theta} = \operatorname{argmax} \sum_{j=1}^{n} \sum_{t=p+1}^{T} \log \left(\int_{\mathbb{R}^k} \prod_{i=1}^{k} \mathbb{1}_{I_{Y_{i,j,t}}} (\lambda_{i,j,t} + x_i) \varphi_R(x) \mathrm{d}x \right),$$

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and

$$\widehat{R}(i_1, i_2) = \operatorname{argmax} \sum_{j=1}^{n} \sum_{t=p+1}^{T} \log \int_{I_{Y_{i_1,j,t}} - \hat{\lambda}_{i_1,j,t}} \Phi\left((2Y_{i_2,j,t} - 1) \frac{\hat{\lambda}_{i_2,j,t} + r x_{i_1}}{\sqrt{1 - r^2}} \right) \varphi(x_{i_1}) \mathrm{d}x_{i_1}.$$

Estimation Results for panel data (2/2)

Proposition 2

Under some reasonable assumptions on the processes $(\zeta_{j,t})_{t\in\mathbb{Z}}$'s:

1) All estimators $\hat{\theta}$, $\hat{\gamma}$ and $\widehat{R}$ are strongly consistent.

2) Moreover, we have the asymptotic normality of

$$\sqrt{n(T-p)}\left(\hat{\theta}-\theta_0\right) \quad \text{and} \quad \sqrt{n(T-p)}\left(\hat{\gamma}-\gamma_0, \widehat{R}-R_0\right).$$

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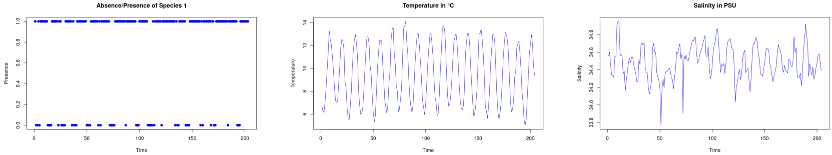

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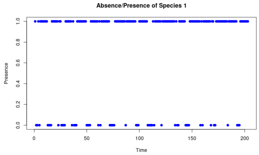

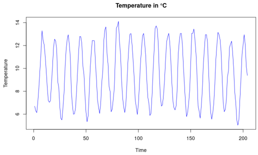

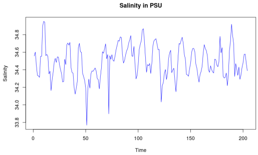

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$$\lambda_t = A \cdot Y_{t-1} + B \cdot X_{t-1},$$

where $(X_t)_{t \in \mathbb{Z}}$ is the process composed by the temperature and salinity.

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$$R = \begin{pmatrix} 1 & 0.2 & -0.5 \\ 0.2 & 1 & -0.3 \\ -0.5 & -0.3 & 1 \end{pmatrix} \quad \text{and} \quad \hat{R} = \begin{pmatrix} 1 & 0.204 & -0.436 \\ 0.204 & 1 & -0.303 \\ -0.436 & 0.204 & 1 \end{pmatrix}.$$

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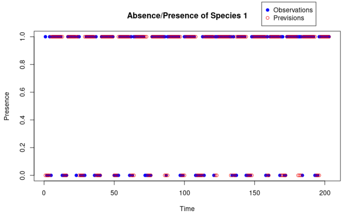

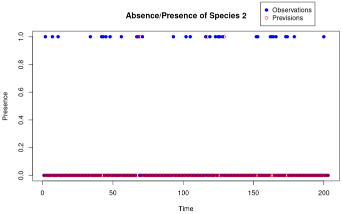

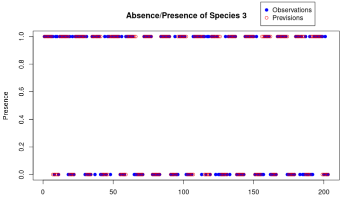

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The data were collected monthly from 1997 to 2013 on 5 different locations in Scotland, and we have access to the covariates: Temperature, Salinity and Oxidised Nitrogen.

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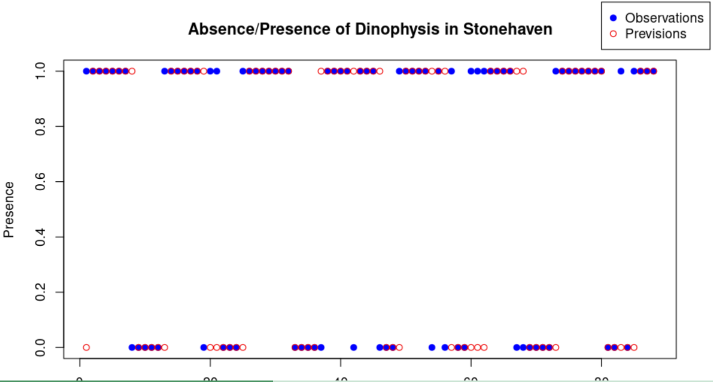

Thank you !