Spectral estimation for noisy Hawkes processes

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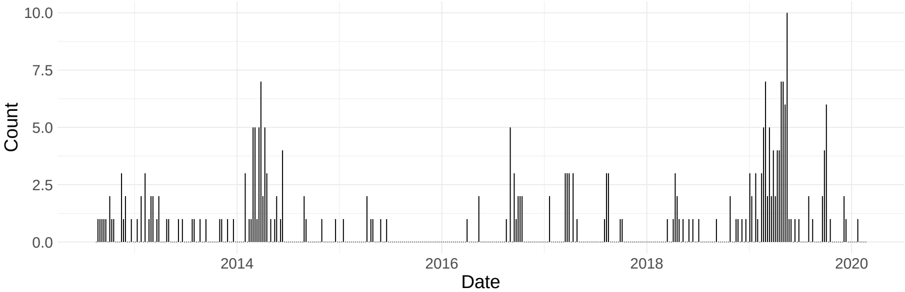
Motivation

Weekly count of measles cases in the prefecture of Tokyo.

- Highly contagious viral disease, transmitting via droplets.
- Sprung back through imported cases and non-vaccinated individuals.
- Notifiable disease: 264 cases in 8 years.



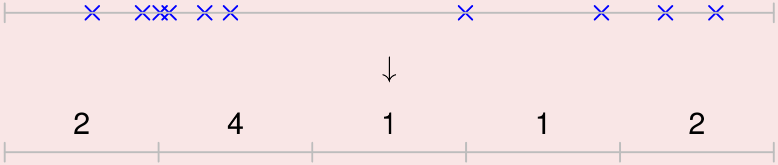
Motivation

Study the dynamics of contagious diseases and their transmission.

- Autoregressive models may be difficult to interpret in an epidemiological context.
- Potentially rarely occurring events.

→ Hawkes process (Meyer, Elias, and Höhle, 2012).

Problem: aggregate datasets



Outline

Outline

1. The Hawkes process
2. A framework for inference from imperfect observation
   - Spectral approach
   - Strong mixing properties for Hawkes processes
3. Identifiability of Hawkes model with Poisson noise

Point process

Definition: Point process N on $\mathbb{R}$

Measurable map N:

$$\begin{aligned} N : (\Omega, \mathcal{F}, \mathbb{P}) &\to (\mathfrak{N}, \mathcal{N}) \\ \omega &\mapsto N(\omega, \cdot) \end{aligned}$$

where $\mathfrak{N}$ is the set of locally finite counting measures on $\mathbb{R}$.

Hawkes process

Conditional intensity λ of point process N

$\lambda(t)dt$ is the conditional probability that there will be an atom of N between t and $t + dt$, given the realisations of N before t:

$$\lambda(t)dt = \mathbb{P}(N(dt) > 0 \mid \{t_j\}, t_j < t)$$

Linear Hawkes process on the real half-line (Hawkes, 1971)

Self-exciting point process defined by its conditional intensity function:

$$\lambda(t) = \eta + \mu \int_{-\infty}^{t} h(t-u) N(\mathrm{d}u) = \eta + \mu \sum_{t_j < t} h(t - t_j)$$

where $\eta > 0$, $\mu \in (0, 1)$, h is an integrable nonnegative function such that $\int_{\mathbb{R}+} h = 1$, and $(t_j)_{j \in \mathbb{N}}$ are realisations of the point process.

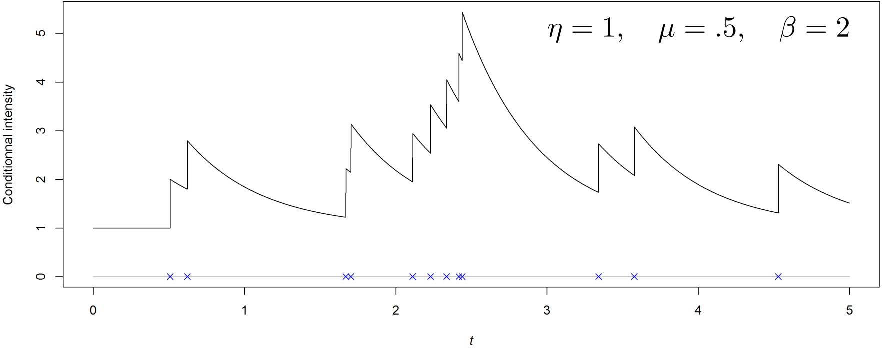
The occurrence of any event increases temporarily the probability of further events occurring.

Hawkes process

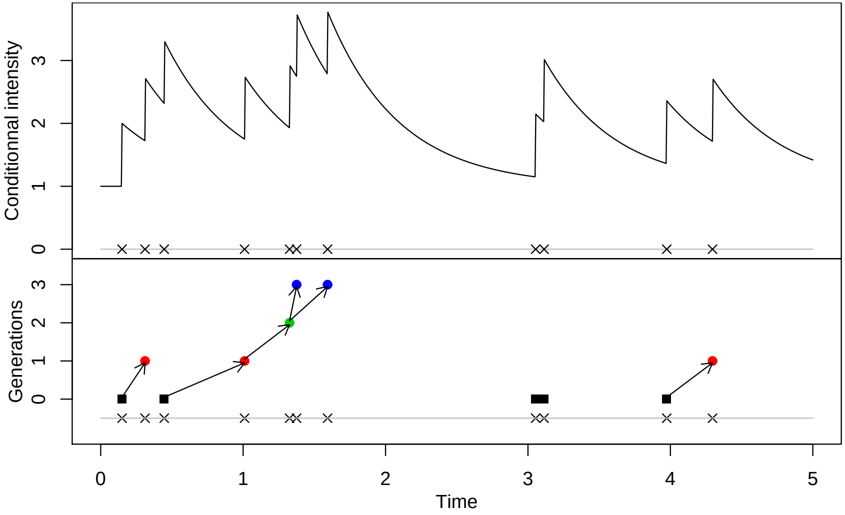
Linear Hawkes process on the real half-line

With exponentially decaying intensity:

$$\lambda(t) = \eta + \mu \sum_{t_j < t} \beta e^{-\beta(t - t_j)}$$



Hawkes process as a branching process



In the literature

- Self-exciting and clustering properties.
- Many disciplines of application:
  - seismology (Adamopoulos, 1976), neurophysiology, finance (Bacry, Mastromatteo, and Muzy, 2015), genomics (Reynaud-Bouret and Schbath, 2010), epidemiology, etc.
  - general review (Reinhart, 2018).
- Interesting properties:
  - Poisson cluster process: each cluster is a continuous-time Galton-Watson tree (Hawkes and Oakes, 1974).
  - Martingale properties of $N(t) - \int_0^t \lambda(s)\mathrm{d}s$ and $\left(N(t) - \int_0^t \lambda(s)\mathrm{d}s\right)^2 - \int_0^t \lambda(s)\mathrm{d}s$.
  - Erlang kernel $\rightarrow$ piecewise deterministic Markov process (Duarte, Löcherbach, and Ost, 2019).

Statistical inference

Parametric estimation of $\theta = (\eta, \mu, h_\phi)$ is usually achieved through **maximum likelihood estimation**,

$$L_T(\theta) = \int_0^T \log \lambda_\theta(t) N(\mathrm{d}t) - \int_0^T \lambda_\theta(t)\mathrm{d}t,$$

or **least-square contrast**,

$$\gamma_T(\theta) = -\frac{2}{T}\int_0^T \lambda_\theta(t) N(\mathrm{d}t) + \frac{1}{T}\int_0^T \lambda_\theta^2(t)\mathrm{d}t.$$

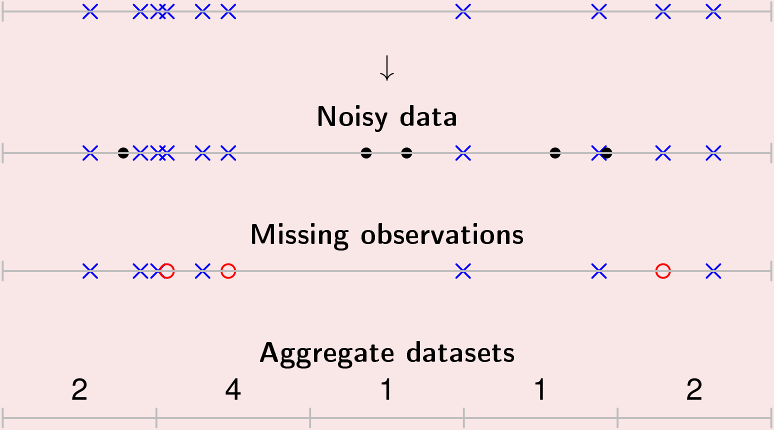
Properties of the estimators

- MLE is consistent and asymptotically Gaussian (Ogata, 1978).
- Oracle results for the LSE (Reynaud-Bouret and Schbath, 2010).

Outline

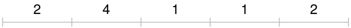
Context

Context: imperfect observation of the process



Problem: The conditional intensity $\lambda(\cdot)$ of the process is either untractable or numerically too expensive to compute.

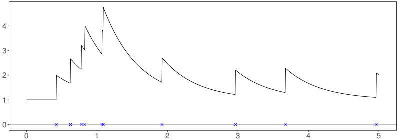
Idea: spectral approach

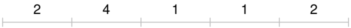


Objective: Estimate $\theta = (\eta, \mu, h_\phi)$ from the count process

Idea: spectral approach

Hawkes process with parameter $\theta = (\eta, \mu, h_\phi)$





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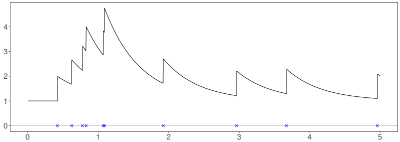
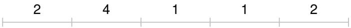


Likelihood of the count process is not tractable



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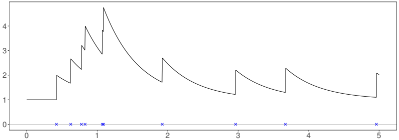
Idea: spectral approach

Time domain

Hawkes process with parameter $\theta = (\eta, \mu, h_\phi)$



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Likelihood of the count process is not tractable

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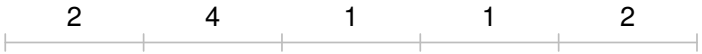


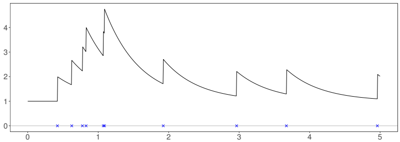
Objective: Estimate $\theta = (\eta, \mu, h_\phi)$ from the count process

Frequency domain

Idea: spectral approach

Time domain

Hawkes process with parameter $\theta = (\eta, \mu, h_\phi)$



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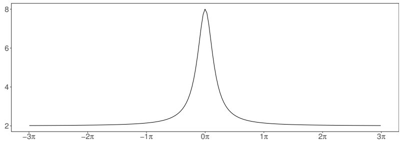
Frequency domain

Bartlett spectrum (Daley and Vere-Jones, 2003, Section 8.2)



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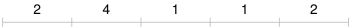

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Time domain

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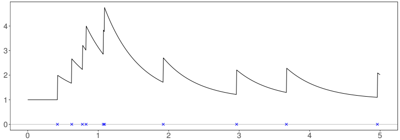

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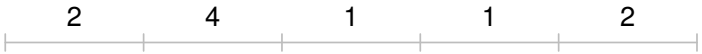

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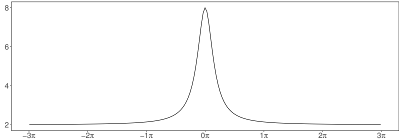


Simple computations

- Spectral density function f_θ;
- Log-spectral likelihood $\mathcal{L}_T(\theta)$;
- Whittle estimator $\widehat{\theta}_T$.

Spectral representation

Bartlett spectrum (Daley and Vere-Jones, 2003, Proposition 8.2.I)

For a second-order stationary point process N on $\mathbb{R}$, then

$$\mathrm{Cov}\left(N(\varphi), N(\psi)\right) = \int_{\mathbb{R}} \widetilde{\varphi}(\omega)\widetilde{\psi^*}(\omega)\Gamma(d\omega)$$

where φ and ψ are functions of rapid decay, $\psi^*(u) = \psi(-u)$, and $\widetilde{\cdot}$ denotes the Fourier transform: $\widetilde{\varphi}(\omega) = \int_{\mathbb{R}} e^{2\pi i \omega u}\varphi(u)du$.
The unique measure $\Gamma(\cdot)$ is called the *Bartlett spectrum* of N.

For the stationary Hawkes process N, the Bartlett spectrum admits a density given by (Daley and Vere-Jones, 2003, Example 8.2(e))

$$\gamma(\omega) = \frac{\eta/(1-\mu)}{|1-\mu\widetilde{h}(\omega)|^2}.$$

The Whittle likelihood (Whittle, 1952)

Consider a bin-count sequence (X_k) with spectral density f_θ. Define

$$\widehat{\theta}_n = \operatorname*{arg\,max}_{\theta \in \Theta} \mathcal{L}_n(\theta)$$

where $\mathcal{L}_n(\theta)$ is the log-spectral likelihood of the process

$$\mathcal{L}_n(\theta) = -\frac{1}{4\pi} \int_{-\pi}^{\pi} \left(\log f_\theta(\omega) + \frac{I_n(\omega)}{f_\theta(\omega)} \right) \mathrm{d}\omega,$$

$I_n(\omega) = \left| \sum_{k=1}^{n} X_k e^{-2\pi i \omega k/n} \right|^2$ is the periodogram of (X_k).

Asymptotic properties for $\widehat{\theta}_n$

Proven for strongly mixing processes (Dzhaparidze, 1986).

Mixing properties for point processes

Define for a process N and $A \in \mathcal{B}(\mathbb{R})$, the cylindrical σ-algebra:

$$\mathcal{E}(A) \coloneqq \sigma\big(\{N \in \mathfrak{N} : N(B) = m\}, B \in \mathcal{B}(A), m \in \mathbb{N}\big).$$

Strong mixing coefficient for a point process N (Westcott, 1972)

Dependence between past and future events:

$$\alpha_N(r) \coloneqq \sup_{t \in \mathbb{R}} \alpha\big(\mathcal{E}_{-\infty}^{t}, \mathcal{E}_{t+r}^{\infty}\big), \qquad \text{where } \mathcal{E}_a^b = \mathcal{E}\big((a, b]\big),$$

where

$$\alpha(\mathcal{A}, \mathcal{B}) \coloneqq \sup\Big\{ \left|\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)\right| : A \in \mathcal{A}, B \in \mathcal{B}\Big\}.$$

For Hawkes processes, ergodicity and mixing results:

- When h has bounded support (Reynaud-Bouret and Roy, 2006; Costa et al., 2020)
- When h is exponential (Graham, 2021; Dion, Lemler, and Löcherbach, 2021).

Strong mixing properties for Hawkes processes

Theorem (Cheysson and Lang, 2022; Boly et al, preprint)

Let N be a stationary Hawkes process on $\mathbb{R}$. Suppose that there exists a $\delta > 0$ such that the reproduction kernel h has a finite moment of order $1 + \delta$, that is

$$\int_{\mathbb{R}} t^{1+\delta} h(t) \mathrm{d}t < \infty.$$

Then N is strongly mixing and

$$\alpha_N(r) = \mathcal{O}\left(r^{-\delta}\right).$$

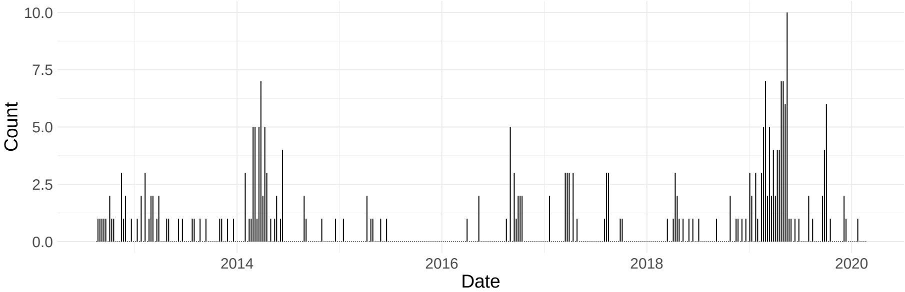
For the bin-count sequence, remark that $\mathcal{F}_a^b \subset \mathcal{E}((a\Delta, (b+1)\Delta])$, then $\alpha_X(r) \leq \alpha_N((r-1)\Delta)$.

Ideas of the proof

$$\alpha_N(r) := \sup_{t\in\mathbb{R}} \alpha\big(\mathcal{E}^t_{-\infty}, \mathcal{E}^\infty_{t+r}\big) = \sup_{t\in\mathbb{R}} \sup_{\substack{\mathcal{A}\in\mathcal{E}^t_{-\infty} \\ \mathcal{B}\in\mathcal{E}^\infty_{t+r}}} \big|\mathrm{Cov}\big(\mathbb{1}_\mathcal{A}(N), \mathbb{1}_\mathcal{B}(N)\big)\big| \quad (1)$$

1. **Control (1) by the covariance of counts.**
2. Downscale to a single branching process by conditioning on the immigrant process.
3. **Control the covariance of counts of a single branching process.**
   - Almost sure extinction of the subcritical Galton-Watson tree;
   - Finite moments for the reproduction kernel.
4. Integrate back w.r.t. the immigrant process.

Case-study: transmission of Measles in Tokyo[1]



Gaussian reproduction kernel: $h(t) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(t-\nu)^2}{2\sigma^2}\right)$

- $\widehat{\nu} = 9.8 \text{ days}, \widehat{\sigma} = 5.9 \text{ days}$

Epidemiology (Centers for Disease Control and Prevention, 2015)

Incubation period: 10-12 days after exposure.

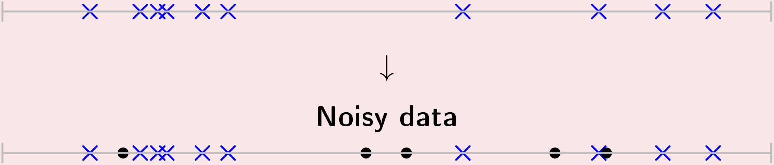
Transmission period: 4 days before to 4 days after rash onset.

[1] https://www.niid.go.jp/niid/en/survaillance-data-table-english.html

Outline

Context

Hawkes process with Poisson noise



Consider a Hawkes process N and an independent Poisson process P with intensity λ_0. Then their superposition M has conditional intensity

$$\lambda_M(t) = \lambda_0 + \eta + \mu \int_{-\infty}^{t} h(t-u) N(\mathrm{d}u).$$

Problem: N cannot be distinguished from P.

Spectral representation and Whittle likelihood

Lemma

Given two independent point processes with Bartlett spectra Γ_1 and Γ_2, their superposition has spectrum $\Gamma_1 + \Gamma_2$.

For the Hawkes process N with Poisson noise P, $\gamma_\theta(\omega) = \gamma_N(\omega) + \lambda_0$, construct the periodogram

$$I_T(\omega) = \frac{1}{T} \left| \int_0^T e^{-2\pi i \omega t} M(\mathrm{d}t) \right|^2 .$$

Define

$$\widehat{\theta}_T = \underset{\theta \in \Theta}{\arg\max}\ \mathcal{L}_T(\theta)$$

where $\mathcal{L}_T(\theta)$ is the log-spectral likelihood (Brillinger, 2012)

$$\mathcal{L}_T(\theta) = -\sum_{k=1}^{\lfloor T \rfloor} \left(\log \gamma_\theta(2\pi k/T) + \frac{I_T(2\pi k/T)}{\gamma_\theta(2\pi k/T)} \right) .$$

The exponential case

- A model $\mathcal{P} = \{P_\theta : \theta \in \Theta\}$ is identifiable if

$$\forall \theta_1, \theta_2 \in \Theta, \qquad P_{\theta_1} = P_{\theta_2} \implies \theta_1 = \theta_2.$$

- In the exponential case, $h(t) = \beta e^{-\beta t}$, $\theta = (\eta, \mu, \beta, \lambda_0)$ and the Bartlett spectrum has density

$$\gamma_\theta(\omega) = \frac{\eta}{1-\mu}\mu\beta^2(2-\mu)\left(\frac{1}{\beta^2(1-\mu)^2 + (2\pi\omega)^2}\right) + \left(\frac{\eta}{1-\mu} + \lambda_0\right).$$

Proposition

The model

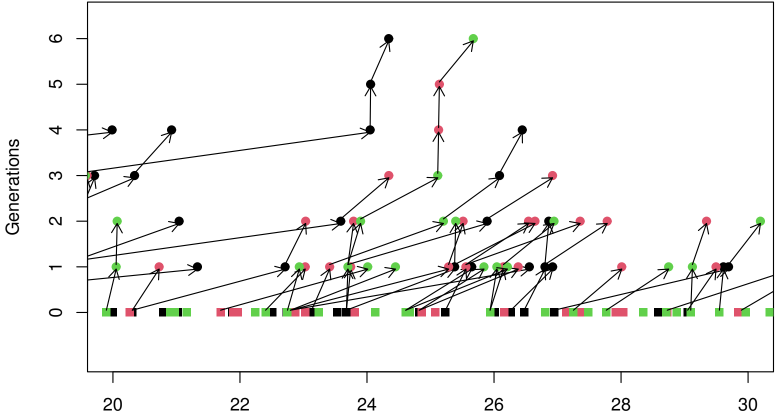
$$\mathcal{Q} = \left\{\gamma_{\eta,\mu,\beta,\lambda_0} : (\eta, \mu, \beta, \lambda_0) \in \mathbb{R}_{\geq 0} \times (0,1) \times R_{\geq 0} \times R_{\geq 0}\right\}$$

is not identifiable. However, if any of the parameter is fixed, then the model is identifiable.

Multivariate Hawkes process

Mutually exciting components

$$\lambda_j(t) = \eta_j + \sum_{i=1}^{d} \mu_{ij} \int_{\mathbb{R}} h_{ij}(t-u) N_i(\mathrm{d}u).$$



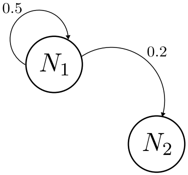
The exponential case: the bivariate Hawkes model

Consider the bivariate exponential Hawkes process with Poisson noise:

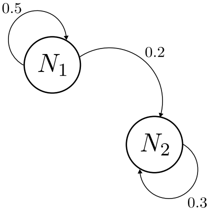
$$\begin{cases} \lambda_1(t) = \lambda_0 + \eta_1 + \int_{-\infty}^{t} \beta_1 e^{-\beta_1(t-u)} \big(\mu_{11} N_1(\mathrm{d}u) + \mu_{21} N_2(\mathrm{d}u)\big), \\ \lambda_2(t) = \lambda_0 + \eta_2 + \int_{-\infty}^{t} \beta_2 e^{-\beta_2(t-u)} \big(\mu_{12} N_1(\mathrm{d}u) + \mu_{22} N_2(\mathrm{d}u)\big). \end{cases}$$

Identifiability

- If $N_1 \perp\!\!\!\perp N_2$, or one is Poisson, then $\mathcal{Q}$ is not identifiable.
- In both scenarii 1 and 2, $\mathcal{Q}$ is identifiable.



Scenario 1



Scenario 2

And for other kernels? The 1d case

- Assume all moments of h exist: $\forall n \geq 1, m_n = \int_{\mathbb{R}} t^n h(t) \mathrm{d}t < \infty$.
- Then $t \mapsto \gamma_\theta(t)$ admits a Taylor expansion around $t \sim 0$

$$\gamma_\theta(t) = \gamma(0) + \sum_{n \geq 1} a_n(\theta) t^{2n},$$

  with $a_n(\theta)$ depending on $m_1, \ldots, m_{2n}$.
- If the Taylor expansion $\theta \mapsto (a_1(\theta), a_2(\theta), \ldots)$ is identifiable, so is the model.

Example: Uniform reproduction kernel $h = \mathbb{1}_{(0,\phi)}$

The model

$$\mathcal{Q} = \left\{\gamma_{\eta,\mu,\phi,\lambda_0} : (\eta, \mu, \phi, \lambda_0) \in \mathbb{R}_{\geq 0} \times (0,1) \times R_{\geq 0} \times R_{\geq 0}\right\}$$

is identifiable.

Thank you for your attention!

For Further Reading I

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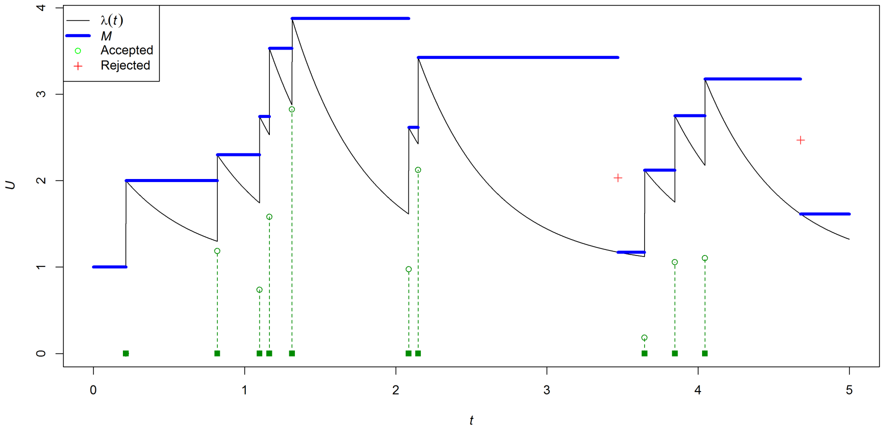
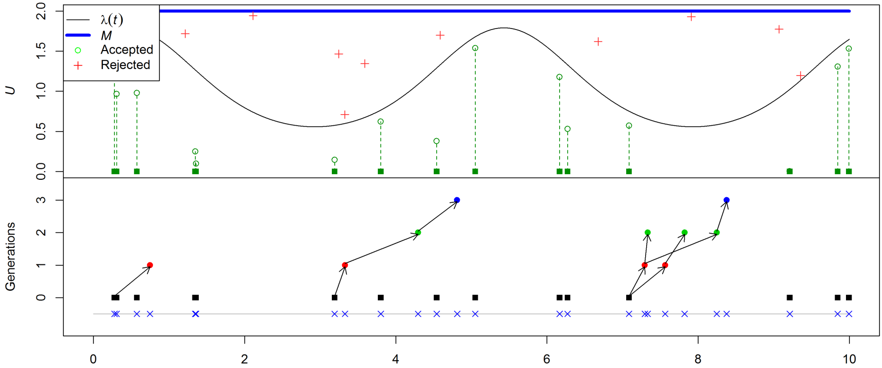
Simulate Hawkes in *R* (Ogata, 1981)

```
sim <- hawkes(T=10, fun=1, repr=1, family="exp", rate=2)
plot(sim, intensity = TRUE)
```



Simulate Hawkes with inhomogeneous background intensity in *R* (Møller and Rasmussen, 2005; Dassios and Zhao, 2013)

```
int <- function(t) exp(.5*cos(2*pi*t/5)+.3*sin(2*pi*t/5))
sim <- hawkes(T=10, fun=int, M=2, repr=1, family="exp", rate=2)
plot(sim$immigrants)
plot(sim)
```



Parameter estimation: residual analysis

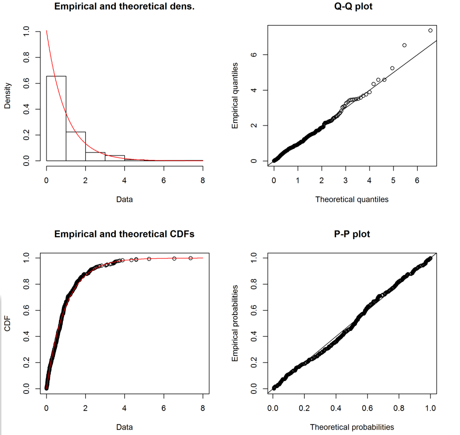
Point process likelihood

Given a point pattern $(t_1, \dots, t_n)$ on an observation interval $[0, T]$, the likelihood function is given by

$$L = \left(\prod_{i=1}^{n} \lambda^*(t_i) \right) \exp\left(- \int_0^T \lambda^*(s) ds \right)$$

Random time change theorem

If $(t_i)_{i \in \mathbb{N}}$ is a point process with conditional intensity $\lambda^*(t_i)$, and $s_i = \int_0^{t_i} \lambda^*(s) ds$, then $(s_i)_{i \in \mathbb{N}}$ is a unit rate Poisson process.



back

Goodness-of-fit diagnostics

- Residual analysis (Ogata, 1988) not available.
- Spectral approach proposed by (Paparoditis, 2000).
- Test based on the distance between the periodogram ordinates and their expected value under the null hypothesis:

$$S_{n,h}(\widehat{\theta}) = nh^{1/2} \int_{-\pi}^{\pi} \left(\frac{1}{nh} \sum_{j=-m}^{m} K\left(\frac{\omega - \omega_j}{h}\right) \left(\frac{I_n(\omega_j)}{f_{\widehat{\theta}}(\omega_j)} - 1\right) \right)^2 \mathrm{d}\omega.$$

Theorem (Paparoditis, 2000, Theorem 2)

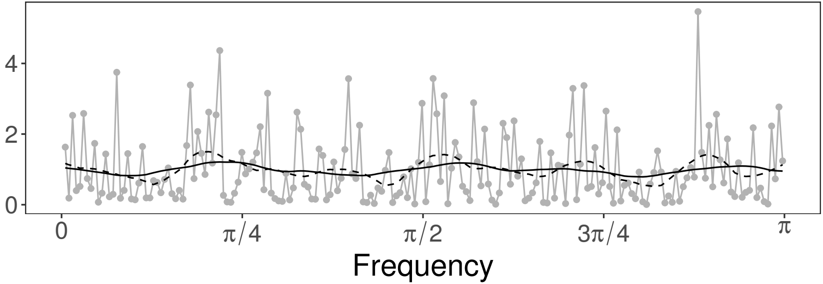
Under some regularity assumptions, for $h \sim n^{-\rho}$ for some $0 < \rho < 1$,

$$S_{n,h}(\widehat{\theta}) - \mu_h \to \mathcal{N}(0, \tau^2),$$

where $\mu_h \propto h, K$ and $\tau^2 \propto K$.

Kernel estimates of the normalised periodogram ordinates:

$$\widehat{q}(\omega, \widehat{\theta}) = \frac{1}{nh} \sum_{j=-m}^{m} K\left(\frac{\omega - \omega_j}{h}\right) \frac{I_n(\omega_j)}{f_{\widehat{\theta}}(\omega_j)}.$$



Bandwidth	Asymptotic p-value	Bootstrap p-value
$h = 0.05$	$p = 0.61$	$p = 0.55$
$h = 0.10$	$p = 0.96$	$p = 0.97$