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Scaling limits of nonlinear functions of random grain model with application to Burgers' equation

Donatas Surgailis (Vilnius University)

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1. Spatial long-range dependence (LRD) and limit theorems (scaling limits)

- Spatial process - stationary random field (RF) $X = \{X(\boldsymbol{t}); \boldsymbol{t} \in \mathbb{R}^d\}$ or $X = \{X(\boldsymbol{t}); \boldsymbol{t} \in \mathbb{Z}^d\}$ with covariance $r_X(\boldsymbol{t}) := \mathrm{Cov}(X(\mathbf{0}), X(\boldsymbol{t}))$

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$$X_\lambda(\phi) \quad := \quad \int_{\mathbb{R}^d} X(\boldsymbol{t})\phi(\boldsymbol{t}/\lambda)\mathrm{d}\boldsymbol{t}, \qquad \text{as } \lambda \to \infty, \tag{1}$$

(or respective sums in the discrete argument case), where $X = \{X(\boldsymbol{t}); \boldsymbol{t} \in \mathbb{R}^d\}$ is a given stationary RF, *for each* ϕ from a class of (test) functions $\Phi = \{\phi : \mathbb{R}^d \to \mathbb{R}\}$.

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- The above approach is common in the theory of generalized RFs
  Gel'fand, I.M., Vilenkin, N.Ya. (1964) *Generalized Functions - Vol.4: Applications of Harmonic Analysis*
  Dobrushin, R.L. (1980) Automodel generalized random fields and their renormgroup. In: R.L. Dobrushin and Ya.G. Sinai (Eds.), *Multicomponent Random Systems*

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- The limit distribution of empirical mean in (3) may be difficult if A *has irregular boundary* ('edge effects')

  Lahiri, S.N. and Robinson, P.M. (2016) Central limit theorems for long range dependent spatial linear processes. Bernoulli 22, 345–375

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- Popular approach in limit theorems for RFs: integration/summation on *rectangles*:

$$\Phi_{\mathrm{rec},d} := \{\mathbb{I}(\boldsymbol{t} \in]\mathbf{0}, \boldsymbol{s}]); \boldsymbol{s} \in \mathbb{R}_+^d\}, \qquad]\mathbf{0}, \boldsymbol{s}] := \prod_{i=1}^{d}]0, s_i]$$

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- Isotropic or uniform scaling $\boldsymbol{t} \to \boldsymbol{t}/\lambda$ in (1) can be replaced by *anisotropic* or *operator scaling* $\boldsymbol{t} \to \lambda^{-\Gamma} \boldsymbol{t}$ where Γ is a $d \times d$-matrix, particularly, a diagonal matrix

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- Operator scaling RF (OSRF):
  Biermé, H., Meerschaert, M.M. and Scheffler, H.P. (2007) Operator scaling stable random fields. Stoch. Process. Appl.

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  We say that *scaling transition occurs at critical* $\frac{\gamma_2}{\gamma_1} = \gamma_0$ (ratio of scaling exponents on different axes of $\mathbb{R}^2$)

- Some ref.:

Puplinskaitė, D. & S.D. (2015) Scaling transition for long-range dependent Gaussian random fields. Stoch. Proc. Appl. 125

Puplinskaitė, D. & S.D. (2016) Aggregation of autoregressive random fields and anisotropic long-range dependence. Bernoulli 22

Pilipauskaitė, V. & S.D. (2017) Scaling transition for nonlinear random fields with long-range dependence. Stoch. Proc. Appl. 127

Biermé, H., Durieu, O. & Wang, Y. (2017) Invariance principles for operator-scaling Gaussian random fields. Ann. Appl. Prob. 27

S.D. (2019) Anisotropic scaling limits of long-range dependent linear random fields on $\mathbb{Z}^3$. J. Math. Anal. Appl. 472

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1. Spatial long-range dependence (LRD) and limit theorems (scaling limits)

- For $d = 1$ and $\phi(t) = \mathbb{I}(t \in]0, s]), s \geq 0$ (5) represent the aggregated sum

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A 'superposition of uniformly scattered in $\mathbb{R}^d$' and randomly dilated grains:

$$X(\boldsymbol{t}) := \sum_{j=1}^{\infty} \mathbb{I}(\boldsymbol{t} \in (\boldsymbol{u}_j + \Xi_j)), \qquad \boldsymbol{t} \in \mathbb{R}^d. \tag{7}$$

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and stochastic integral representation

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- Closely related object: the (random) *Boolean set*:

$$\mathcal{X} := \bigcup_{j=1}^{\infty}(\boldsymbol{u}_j + R_j^{1/d}\Xi^0) \quad \subset \mathbb{R}^d. \tag{9}$$

- $Leb_d(\Xi_j) = R_j Leb_d(\Xi^0)$: dilates $Leb_d(\Xi^0)$ by random factor R_j
- $X(\boldsymbol{t})$ in (7) counts the number of random grains which cover $\boldsymbol{t} \in \mathbb{R}^d$
- $X(\boldsymbol{t})$ has marginal Poisson distribution with mean

$$\mu = \mathrm{E}X(\boldsymbol{t}) = \int_{\mathbb{R}^d} \mathrm{P}(\boldsymbol{t} - \boldsymbol{u} \in R^{1/d}\Xi^0)\mathrm{d}\boldsymbol{u} = Leb_d(\Xi^0)\mathrm{E}R < \infty$$

and stochastic integral representation

$$X(\boldsymbol{t}) = \int_{\mathbb{R}^d \times \mathbb{R}_+} \mathbb{I}(\boldsymbol{t} - \boldsymbol{u} \in r^{1/d}\Xi^0)\mathcal{N}(\mathrm{d}\boldsymbol{u}, \mathrm{d}r), \quad \boldsymbol{t} \in \mathbb{R}^d, \tag{8}$$

$\mathcal{N}(\mathrm{d}\boldsymbol{u}, \mathrm{d}r)$: Poisson random measure with $\mathrm{E}\mathcal{N}(\mathrm{d}\boldsymbol{u}, \mathrm{d}r) = \mathrm{d}\boldsymbol{u}F(\mathrm{d}r)$

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- Boolean model is basic in stochastic geometry and stereology

2. Random grain (RG) model

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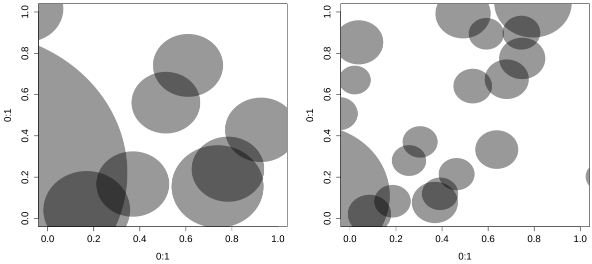
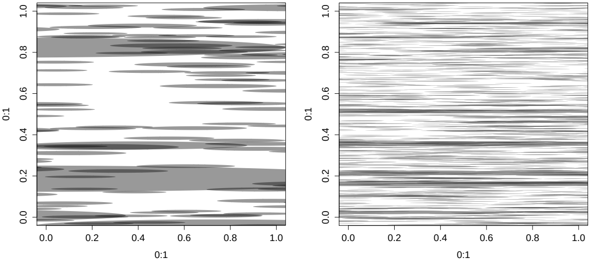
- Trajectories of RG model very different from Gaussian:



Isotropically scaled random ball model, $\gamma = 1, \alpha = 3/2$. Left: $\lambda = 5$, right: $\lambda = 10$



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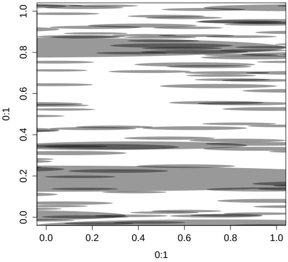

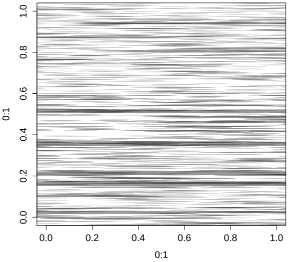

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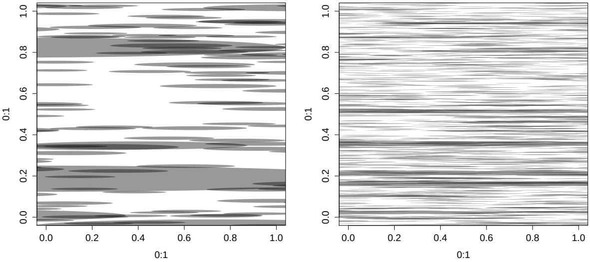

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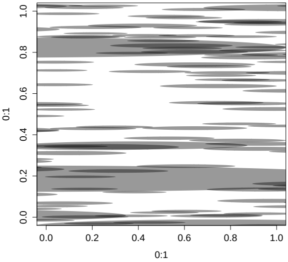

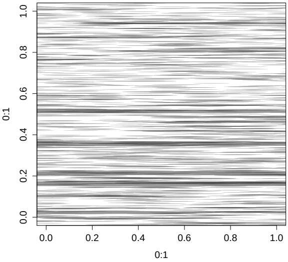

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Well-known: RG is LRD [= nonintegrable covariance] if $P(R > r) = F(r, \infty)$ *varies regularly at* ∞ *with exponent* $\alpha \in (1, 2)$.

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$$r_X(\boldsymbol{t}) \quad \sim \quad \|\boldsymbol{t}\|^{-d(\alpha-1)}\ell(\frac{\boldsymbol{t}}{\|\boldsymbol{t}\|}), \qquad |\boldsymbol{t}| \to \infty, \quad 1 < \alpha < 2, \tag{11}$$

where $\ell(\boldsymbol{z}), \|\boldsymbol{z}\| = 1$ is a bdd cont. (angular) function

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$$c(\phi) := \int_{\mathbb{R}^{2d}} \phi(\boldsymbol{t}_1)\phi(\boldsymbol{t}_2)\ell\Big(\frac{\boldsymbol{t}_1 - \boldsymbol{t}_2}{\|\boldsymbol{t}_1 - \boldsymbol{t}_2\|}\Big)\frac{\mathrm{d}\boldsymbol{t}_1\mathrm{d}\boldsymbol{t}_2}{\|\boldsymbol{t}_1 - \boldsymbol{t}_2\|^{d(\alpha-1)}}$$

2. Random grain (RG) model ([KLNS] = Kaj et al (2007))

- Scaling limits with aggregation for sums $X_{\lambda,M}(\phi) := \sum_{j=1}^{M} \int_{\mathbb{R}^d} X_j(\boldsymbol{t})\phi(\boldsymbol{t}/\lambda)\mathrm{d}\boldsymbol{t}$ of independent RG models.

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2. Random grain (RG) model ([KLNS] = Kaj et al (2007))

Theorem (1)

Let Assumption LRD hold, $M = \lambda^{\gamma}$ $(\gamma > 0)$. Then for any $\phi \in \Phi$

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- $B_{\alpha}(\phi)$: *Gaussian RF* with $\mathrm{Var}(B_{\lambda}(\phi)) = c(\phi)$ in (12). It is represented as integral $B_{\alpha}(\phi) = \int_{\mathbb{R}^d \times \mathbb{R}_+} W_{\alpha}(\mathrm{d}\boldsymbol{u}, \mathrm{d}r) \int_{\mathbb{R}^d} \phi(\boldsymbol{t}) \mathbb{I}(\boldsymbol{t} - \boldsymbol{u} \in r^{1/d}\Xi^0)\mathrm{d}\boldsymbol{t}$ w.r.t. *Gaussian* noise with $\mathrm{E}W_{\alpha}(\mathrm{d}\boldsymbol{u}, \mathrm{d}r)^2 = c_f r^{-1-\alpha}\mathrm{d}r\mathrm{d}\boldsymbol{u}$
- $L_{\alpha}(\phi)$: *α-stable RF* written as stochastic integral $L_{\alpha}(\phi) = \int_{\mathbb{R}^d} \phi(\boldsymbol{t}) L_{\alpha}(\mathrm{d}\boldsymbol{t})$ w.r.t. α-stable random measure L_{α}
- $J_{\alpha}(\phi)$: *'intermediate Poisson' RF* written as stochastic integral $J_{\alpha}(\phi) = \int_{\mathbb{R}^d \times \mathbb{R}_+} \tilde{N}_{\alpha}(\mathrm{d}\boldsymbol{u}, \mathrm{d}r) \int_{\mathbb{R}^d} \phi(\boldsymbol{t}) \mathbb{I}(\boldsymbol{t} - \boldsymbol{u} \in r^{1/d}\Xi^0)\mathrm{d}\boldsymbol{t}$ w.r.t. *centered Poisson* random measure with variance $\mathrm{E}\tilde{N}_{\alpha}(\mathrm{d}\boldsymbol{u}, \mathrm{d}r)^2 = c_f r^{-1-\alpha}\mathrm{d}r\mathrm{d}\boldsymbol{u}$ (the same as W_{α})

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$$X_M(\phi) \approx \sum_j Leb_d((\boldsymbol{u}_{j,M} + R_j \Xi^0) \cap]0, \lambda]^d),$$

  where $\{\boldsymbol{u}_{j,M}\}$ is Poisson process with intensity $M \mathrm{d}\boldsymbol{u} = \lambda^\gamma \mathrm{d}\boldsymbol{u}$

- Intuition (ctnd): If random grain $\boldsymbol{u}_{j,M} + R_j \Xi^0 \subset]0, \lambda]^d$ then the corresponding $Leb_d((\boldsymbol{u}_{j,M} + R_j^{1/d} \Xi^0) \cap]0, \lambda]^d) = Leb_d(R_j^{1/d} \Xi^0) \propto R_j$ which are independent and α-tailed r.v.

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- In our case $n = \lambda^{d+\gamma}, c_n = \lambda^d, c_n/n^{1/\alpha} = \lambda^{d - \frac{d+\gamma}{\alpha}}$ and $d - \frac{d+\gamma}{\alpha} = 0$ is equivalent to $\gamma = \gamma_0 = d(\alpha - 1)$ exactly as in the above theorem.

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- Hermite polynomials $H_k(x), x \in \mathbb{R}, k \in \mathbb{N}$ related $Z \sim N(0,1)$ are defined by power series

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(i) (Orthogonality property): For any $k, \ell \in \mathbb{N}$

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Examples of random processes with multivariate Poisson distribution: random grain model, *trawl process with Poisson seed*:

Barndorff-Nielsen, O.E., Lunde, A., Shepard, N. & Veraart, A.E.D. (2014) Integer-valued trawl processes: a class of stationary infinitely divisible processes. Scand. J. Statist. 41, 693–724.

Doukhan, P., Jakubowski, A., Lopes, S.R.C. & S.D. (2019) Discrete-time trawl processes. Stoch. Proc. Appl. 129

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S.D. (1984) On multiple Poisson stochastic integrals and associated Markov semigroups. Probab. Math. Statist. 3

Proof of Lemma 1. (i) Use generating function $\mathcal{P}(u; x, \mu) = (1+u)^x \mathrm{e}^{-u\mu}$ of Charlier polynomials in (16):

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(ii) Immediate from (i) and (17).

Proof of Lemma 1. (i) Use generating function $\mathcal{P}(u;x,\mu) = (1+u)^x \mathrm{e}^{-u\mu}$ of Charlier polynomials in (16):

$$\begin{aligned}
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&= \mathrm{e}^{uv\mu_3} = \sum_{k=0}^{\infty}\frac{(uv\mu_3)^k}{k!}.
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(iii) Apply (ii) to $G_1(x) := \mathbb{I}(x=n)$, $G_2(x) := \mathbb{I}(x=m)$, for given $n, m \in \mathbb{N}$. By (19), (16), $c_{G_1}(k) = \mathrm{E}[D_+^k \mathbb{I}(N_1 = n)] = D_-^k p(n;\mu) = (-1)^k \mu^{-k} P_k(n;\mu)p(n;\mu)$, $c_{G_2}(k) = \mathrm{E}[D_+^k \mathbb{I}(N_2 = m)] = D_-^k p(m;\mu) = (-1)^k \mu^{-k} P_k(m;\mu)p(n;\mu)$, yielding (iii).

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Let $G_i, N_i, i = 1, 2$ *be as in Lemma 1,* $k_C^*(G_i)$ = *Charlier rank of* G_i. *Then*

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where $k_C^* := k_C^*(G_1) \vee k_C^*(G_2)$ *and*

$$|R(k_C^*)| \le \frac{(\mu_3/\mu)^{k_C^*+1}}{1-(\mu_3/\mu)} \prod_{i=1}^{2} \mathrm{E}^{1/2} G(N_i)^2. \tag{22}$$

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$$X_M(\boldsymbol{t}) = \int_{\mathbb{R}^d \times \mathbb{R}_+} \mathbb{I}(\boldsymbol{t} - \boldsymbol{u} \in r^{1/d} \Xi^0) \mathcal{N}_M(\mathrm{d}\boldsymbol{u}, \mathrm{d}r)$$

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- $h_{G,\mu}(1) = \mu^{-1}\mathrm{E}G(Z_\mu)Z_\mu$

4. Scaling of nonlinear functions of RG model

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Let $M = \lambda^\gamma$ for some $\gamma > 0$. Then for any $\phi \in \Phi$ as $\lambda \to \infty$

$$\lambda^{(\gamma/2)-H(\gamma)}(Y_{\lambda,M}(\phi) - \mathrm{E}Y_{\lambda,M}(\phi)) \xrightarrow{\mathrm{d}} h_{G,\mu}(1) \begin{cases} B_\alpha(\phi), & \gamma > d(\alpha-1), \\ L_\alpha(\phi), & \gamma < d(\alpha-1), \\ J_\alpha(\phi), & \gamma = d(\alpha-1), \end{cases}$$

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$$\lambda^{-d/\alpha}(Y_\lambda(\phi) - \mathrm{E}Y_\lambda(\phi)) \ \xrightarrow{\mathrm{d}}\ c_{G,\mu}(1)L_\alpha(\phi),$$

where $c_{G,\mu}(1) = \mathrm{E}G(X(\mathbf{0}))(X(\mathbf{0}) - \mathrm{E}X(\mathbf{0}))$ is the first Charlier coefficient of G and $L_\alpha(\phi)$ is the same α-stable RF as in part 1.

Example (Boolean model)

The Boolean model $\hat{X}(\boldsymbol{t}) = X(\boldsymbol{t}) \wedge 1$ corresponds to $Y(\boldsymbol{t}) = G(X(\boldsymbol{t}))$ with $G(x) = x \wedge 1, x \in \mathbb{N}$.

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Charlier coefficients: $c_{G,\mu}(0) = 1 - \mathrm{e}^{-\mu}, c_{G,\mu}(k) = (-1)^{k+1}\mathrm{e}^{-\mu}\,(k \geq 1)$

Example (Boolean model)

The Boolean model $\hat{X}(\boldsymbol{t}) = X(\boldsymbol{t}) \wedge 1$ corresponds to $Y(\boldsymbol{t}) = G(X(\boldsymbol{t}))$ with $G(x) = x \wedge 1, x \in \mathbb{N}$.

For $\phi(\boldsymbol{x}) = \mathbb{I}(x \in A)$, $Y_\lambda(\phi) = Leb_d(\mathcal{X} \cap \lambda A) =: \hat{X}_\lambda(A)$ (= volume of $\{X(\boldsymbol{t}) = 1\} \cap \lambda A$)

Charlier coefficients: $c_{G,\mu}(0) = 1 - \mathrm{e}^{-\mu}, c_{G,\mu}(k) = (-1)^{k+1}\mathrm{e}^{-\mu}\ (k \geq 1)$

Corollary (1)

Let $A \subset \mathbb{R}^d$ be a bounded Borel set and $X(\boldsymbol{t})$ RG model as in Thm 1. Then

$$\lambda^{-d/\alpha}(\hat{X}_\lambda(A) - \mathrm{E}\hat{X}_\lambda(A)) \xrightarrow{\mathrm{d}} \mathrm{e}^{-\mu} L_\alpha(A), \quad \lambda \to \infty$$

where $L_\alpha(A)$ is asymmetric α-stable r.v. with
$\mathrm{E}\mathrm{e}^{\mathrm{i}\theta L_\alpha(A)} = \exp\{-\sigma_\alpha|\theta|^\alpha Leb_d(A)(1 - \mathrm{i}\,\mathrm{sgn}(\theta)\tan(\pi\alpha/2))\}, \theta \in \mathbb{R}$.

4. Scaling of nonlinear functions of RG model

Example (Exponential model)

$$\mathcal{E}_M(\boldsymbol{t}) := \mathrm{e}^{a(X_M(\boldsymbol{t}) - \mathrm{E}X_M(\boldsymbol{t}))/M^{1/2}}, \quad \mathcal{E}_{\lambda,M}(\phi) := \int_{\mathbb{R}^d} \phi(\boldsymbol{t}/\lambda)\mathcal{E}_M(\boldsymbol{t})\mathrm{d}\boldsymbol{t}.$$

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Particular case of (23) corresponding to $G(x) = \mathrm{e}^{ax}$. Note $D_+^k G(x) = (\mathrm{e}^a - 1)^k \mathrm{e}^{ax}$ and $c_{G,\mu}(k) = (\mathrm{e}^a - 1)^k \mathrm{e}^{(\mathrm{e}^a - 1)\mu}$, $k \in \mathbb{N}$. We also have

$$\begin{aligned} M^{1/2} c_{G(\cdot/M^{1/2}),\mu M}(1) &= \exp\{(\mathrm{e}^{a/M^{1/2}} - 1 - (a/M^{1/2}))\mu M\} M^{1/2}(\mathrm{e}^{a/M^{1/2}} - 1) \\ &\to a\mathrm{e}^{a^2\mu/2} = \mathrm{E}[\mathrm{e}^{aZ_\mu} Z_\mu] = h_{G,\mu}(1) \end{aligned}$$

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Corollary (2)

Let $X_M(\boldsymbol{t})$ be as in Thm 1 and $G(x) = \mathrm{e}^{ax}$. Then

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Corollary (2)

Let $X_M(\boldsymbol{t})$ be as in Thm 1 and $G(x) = \mathrm{e}^{ax}$. Then

$$\lambda^{(\gamma/2) - H(\gamma)}(\mathcal{E}_{\lambda,M} - \mathrm{E}\mathcal{E}_{\lambda,M}(\phi)) \quad \xrightarrow{\mathrm{d}} \quad a\mathrm{e}^{a^2\mu/2} \begin{cases} B_\alpha(\phi), & \gamma > d(\alpha - 1), \\ L_\alpha(\phi), & \gamma < d(\alpha - 1), \\ J_\alpha(\phi), & \gamma = d(\alpha - 1), \end{cases}$$

where $H(\gamma), B_\alpha(\phi), L_\alpha(\phi), J_\alpha(\phi)$ are the same as in Thm 1.

Burgers' equation with (random) potential initial data:

$$\begin{aligned} \partial\vec{v}(t,\boldsymbol{x})/\partial t + (\vec{v}(t,\boldsymbol{x}),\nabla)\vec{v}(t,\boldsymbol{x}) &= \frac{1}{2}\kappa\Delta\vec{v}(t,\boldsymbol{x}), \quad t>0, \boldsymbol{x}\in\mathbb{R}^d \\ \vec{v}(0,\boldsymbol{x}) &= -\nabla\xi(\boldsymbol{x}), \end{aligned}$$

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- behavior of $\vec{v}(t,\boldsymbol{x})$ presents considerable physical and mathematical interest and has been extensively studied
- M. Rosenblatt, Ya. Sinai, S. Molchanov, W. Woyczynski, N. Leonenko, ...

- Hopf-Cole substitution:

$$\vec{v}(t,\boldsymbol{x}) = -\kappa \nabla \log u(t,\boldsymbol{x}) = -\frac{\kappa \nabla u(t,\boldsymbol{x})}{u(t,\boldsymbol{x})}$$

  with scalar-valued $u(t,\boldsymbol{x})$ satisfying *heat equation*
  $\partial u(t,\boldsymbol{x})/\partial t = \frac{1}{2}\kappa \Delta u(t,\boldsymbol{x})$
  with the *exponential* initial condition $u(0+,\boldsymbol{x}) = \exp\{\xi(\boldsymbol{x})/\kappa\}, \boldsymbol{x} \in \mathbb{R}^d$.

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$g(t,\boldsymbol{x},\boldsymbol{y}) := (2\pi\kappa t)^{-d/2}\exp\{-\|\boldsymbol{x}-\boldsymbol{y}\|^2/2\kappa t\}$:

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- integrals in numerator and denominator resemble $Y_\lambda(\phi) = \int_{\mathbb{R}^d} G(\xi(\boldsymbol{y}))\phi(\boldsymbol{y}/\lambda)\mathrm{d}\boldsymbol{y}$
  with $G(x) = \mathrm{e}^{x/\kappa}$, $\phi(\boldsymbol{y}) = \nabla g(t, \boldsymbol{x}, \boldsymbol{y})$ and $\phi(\boldsymbol{y}) = g(t, \boldsymbol{x}, \boldsymbol{y})$

- For $\kappa > 0$ fixed the limit distribution of $\vec{v}_\lambda(t, \boldsymbol{x})$ was studied for several models of initial RF $\xi = \{\xi(\boldsymbol{y}), \boldsymbol{y} \in \mathbb{R}^d\}$ with short and long range dependence [Gaussian, Gaussian subordinated, shot-noise, Cox]

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- This talk: initial potential RF = aggregated RG model

$$\xi_M(\boldsymbol{y}) := M^{-1/2}(X_M(\boldsymbol{y}) - \mathrm{E}X_M(\boldsymbol{y})), \quad \boldsymbol{y} \in \mathbb{R}^d, \tag{24}$$

with intensity $M = \lambda^\gamma$ increasing with λ for some $\gamma > 0$

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- The meaning of intial condition $\vec{v}(0+, \boldsymbol{x}) = -\nabla \xi_M(\boldsymbol{x})$ ignored

Thm 2 and Example 2 lead to

Corollary

1. Let $\vec{v}_\lambda(t, \boldsymbol{x})$ *be as in* (23), (24), *with* X_M, M *satisfying the conditions of Thm 1.*

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$$\lambda^{1+d+\frac{\gamma}{2}-H(\gamma)}\vec{v}_\lambda(t, \boldsymbol{x}) \quad \xrightarrow{\text{fdd}} \quad \begin{cases} B_\alpha(\nabla g(t, \boldsymbol{x}, \cdot)), & \gamma > d(\alpha - 1), \\ L_\alpha(\nabla g(t, \boldsymbol{x}, \cdot)), & \gamma < d(\alpha - 1), \\ J_\alpha(\nabla g(t, \boldsymbol{x}, \cdot)), & \gamma = d(\alpha - 1), \end{cases} \tag{25}$$

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where $H(\gamma)$ and the limit RFs are the same as in Thms 1-2.

Thm 2 and Example 2 lead to

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2. $\vec{v}_\lambda(t, \boldsymbol{x})$ be as in (23) with $\xi(\boldsymbol{y}) = X(\boldsymbol{y})$ given in (8) ($M = 1$). Then, as $\lambda \to \infty$

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where L_α is α-stable RF as in part 1.

6. Open questions

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3. Cox RG model: Poisson grains with *random intensity*